STUDENTS' UNDERSTANDING OF LOGARITHMIC FUNCTION NOTATION

Rachael Kenney
North Carolina State University
rhkenney@unity.ncsu.edu

The purpose of this study is to investigate how college students interpret logarithmic notation and how these interpretations inform students' understandings of rules for working with logarithmic equations. The framework used for this study is the procept theory of Gray and Tall (1994), which proposes that the use of mathematical symbols enables mathematical concepts to be treated as both a process and an object at the same time, and that it is the ability to deal with this dual nature of notation that separates the less able math student from the more able. The results suggest that most students in the study lack a process-object understanding of logarithms.

Purpose and Background

Students struggle greatly with both the concept of logarithms as inverse functions and the processes and procedures needed for working with logarithmic equations. Much of this difficulty stems from trouble students have interpreting notation used to express logarithms. What is it that students are "seeing" when working with logarithmic and exponential functions? What meanings are they constructing about the symbols involved? To be successful, students must be able to interpret the symbols used as both an expression of the object of a logarithm and an indication of the process needed to work with the function (Gray & Tall, 1994; Kinzel, 1999; Sajka, 2003; Weber, 2002b). The term *process* here means a sequence of steps that a student might perform to execute a mathematical problem. This paper investigates the idea that notation used in logarithmic and exponential functions is often misunderstood by students and becomes a hindrance to their conceptual understanding of log functions (Stacey & MacGregor, 1997).

Little research in math education has looked specifically at students' understanding of logarithms (Weber, 2002a; Weber, 2002b). However, researchers such as Dubinsky & Harel (1992), Kinzel (1999), Sajka (2003), and Tall et al (2000) have taken a close look at students' understandings of general function notation. Researchers suggest that the dual nature of the symbols, serving as both an indication of a particular operation and an object upon which to be operated, can be difficult for students to interpret (Kinzel; Gray & Tall, 1994, Sajka). For example, an expression such as $f(x) = 3x+1$ can be interpreted as a rule for a procedure, or as an object that can be manipulated (Kinzel). Students must be able to understand the context clues in mathematical problems to decide which interpretation to follow, which can be a difficult task (Sajka). Both Stacey & MacGregor (1997) and Kinzel have observed that students have a limited understanding of algebraic symbols used in different contexts and the ways that they are used to communicate in mathematics.

According to Gray and Tall (1994), mathematicians cope with an object's dual nature as a process and an object by giving it the same notation. For example, they use *a*/*b* to represent both the process of division and the object of a fraction. This dual use of notation enables advanced mathematical thinkers to deal with the duality of object and process, but less-able learners are left unable to comprehend this duality and, instead, remain focused only on a process understanding of functions and their notation (Gray & Tall). They rely on memorized procedures evoked by certain notation. A *procedure* here means a specific algorithm used to

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005).
Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

implement a process (Gray & Tall), and *procedural thinking* is defined as strict attention to procedures and the algorithmic tools that support them (Tall et al, 2000). Procedural thinking on a problem may allow students to do specific computations correctly without understanding why their algorithms work; however, students will encounter difficulties as they try to build new ideas on their procedural understanding (Tall et al). This is particularly true when learning logarithmic functions, because until this point in their learning of functions, students have been presented with notation that gives a clear rule for what to do with an input value (Hurwitz, 1999). For example, $f(x) = 2x+3$ means to double the input and add three. Logarithmic functions cannot be considered in the same fashion. Understanding logarithmic functions relies on being able to interpret the notation and symbols involved. The definition of a logarithmic function in many textbooks is given as follows:

$$\log_a (x) = y \text{ if and only if } a^y = x.$$

Students must be able to understand this notation as showing both a process and an object to successfully work with the function (Weber, 2002b). The logarithmic notation is used here as both a referent to a specific logarithmic function, and as an indicator of the value needed to be used in an exponentiation process. The value x is both an input value for the logarithmic function and the product of y factors of a (Weber, 2002b). To solve both exponential and logarithmic equations, students must be able to understand connections between the logarithmic and exponential forms and be able to combine and reverse the processes involved in both forms (Dubinsky & Harel, 1992; Weber, 2002a). However, Hurwitz finds that logarithmic notation leaves students "bereft of a succinct way to verbalize the operation performed on the input" (p. 344), and that the change from the familiar $f(x)$ makes it difficult for students to interpret the logarithm as a function output. The question to explore, then, is what do students understand when they see logarithmic notation? Do students interpret logarithmic functions as representing both an object and a process? How does their interpretation guide their understandings of rules for manipulating logarithmic expressions and working with logarithmic functions and equations?

Theoretical Framework

The phenomenon that this study intends to explain is the way in which students understand the concept of logarithms and use these understandings to solve problems that involve logarithms. The intention is not to look specifically at the learning process, but rather the understanding and conceptions that have been formed as a result of learning. The framework used for this study is the procept theory formulated by Gray and Tall (1994), which proposes that the use of mathematical symbols enables students to consider mathematical concepts to be both a process and an object at the same time, and that it is the ability to deal with this dual nature of notation that separates the less able math student from the more able.

In this framework, symbols act as a pivot between being able to think of a symbol as a process and as an object (Tall et al, 2000). Gray and Tall (1994) refer to this notion of a symbol representing both a process and a concept resulting from that process as a *procept*. They define an elementary procept as consisting of three components: a process, an object produced by the process, and a symbol representing the process or the object. Procepts, then, consist of several elementary procepts that all have the same object. *Proceptual thinking* is defined as an "ability to compress stages in symbol manipulation to the point where symbols are viewed as objects that can be decomposed and recomposed in flexible ways" (Gray & Tall, p. 132). In other words, students who think proceptually have the flexibility to see symbols as an indicator for carrying out a procedure, or as a compact representation of an object that can be manipulated and acted upon. For logarithms, an example of an elementary procept might be the symbol log(x), in the equation $\log(x) + \log(x - 9) = 1$. The symbol represents an object that can be decomposed and recomposed into a new object according to the properties of logarithms, and also indicates the process of exponentiation. This study examines the extent to which students demonstrate proceptual thinking when interpreting notation to solve such problems. Under this framework, the researcher tries to determine if, as Gray and Tall suggest, less able students are not simply slower learners but are, in fact, developing different techniques for problem solving due to their interpretations of the symbols.

Methods

The subjects for this study were first-year college students in two different pre-calculus classes, MA-107 and MA-111, taught by the researcher at a university. Most of the students in both classes were between the ages of 17 and 20. Many of the students were weak in their understanding of basic algebra skills.

There were two phases of data collection for this study. Phase I involved a five-problem questionnaire administered to 59 pre-calculus students enrolled in MA-111. The questionnaire was voluntary, and no points were offered toward the class grade for participation. The purpose of the questionnaire was to give the researcher some understanding of the different ways in which students interpret and manipulate logarithmic functions. Two of the five questions were used in the analysis. Data gained from this questionnaire was used in the design of the second research tool. However, due to time constraints on the researcher and the students, no students from MA-111 were able to participate in the second part of the study.

Phase II of the data collection was a three part task-based interview conducted with two students from MA-107. These students had just finished a lesson on exponential and logarithmic functions and had been recently tested on the material. The two girls interviewed, Anna and Lynn, were chosen by the researcher because: 1) they were active learners who asked questions and discussed problem solving strategies when doing group work in class, and 2) they had used incorrect methods to solve the equation $\log(x) + \log(x+9) = 1$ on their tests. Questions were chosen by the researcher to try to determine what sense the students were making of different logarithmic forms and their interpretations of the notation.

Results

Phase I

The questionnaires provided some interesting insights into students' understanding of logarithms. The first question contained five pairs of logarithmic expressions in which students were asked to circle whether the pairs were equivalent or not and to explain their choice in a sentence (see Figure 1). Almost all students reasoned correctly that the expressions in question *a*

a.	$\log_3(2)$	= or ≠ (circle one)	$\log_4(2)$
b.	$\log_a(1)$	= or ≠ (circle one)	$\log_b(1)$
c.	$\log_3(x) + \log_3(x+1)$	= or ≠ (circle one)	$\log_5(x) + \log_5(x+1)$
d.	$\ln(x)$	= or ≠ (circle one)	$\log_{10}(x)$

Figure 1. Paired expressions problems from the questionnaire

were not equal because the bases were different, but 26 of the 59 students used the same argument to say that the expressions in *b* were not equal. The notation here did not seem to suggest a process or object, perhaps due to the generic values for the bases. Students saw no meaningful relationship in the log symbols that supported mathematical activity (Kinzel, 1999). A similar lack of procedural and conceptual understanding of logarithms was exhibited in question *c*. Here, 19 out of 59 students believed the expressions were equivalent, most reasoning that log was irrelevant because it could be "cancelled out". The logarithmic notation obviously had no meaning for these students.

Another surprising result was that 35 out of 59 students answered on question *d* that $\ln(x)$ was equivalent to $\log(x)$. In these classes, $\ln(x)$ is notation for log base *e*, and $\log(x)$ notates log base 10. The special notation was presented and reinforced in class; however, a majority of students misinterpreted it. According to Kinzel (1999), teachers must be explicit about what is being represented by mathematical symbols if they are to be understood and used properly. Perhaps the students' misconceptions came from insufficient explicit teaching of this concept, but it could also be that the change from *log* to *ln* caused major problems for students and prevented them from forming a process or concept view of $\ln(x)$.

The second problem on the questionnaire required students to solve for *x* given the equation $\log(x) + \log(x+9) = 1$. Students were asked to show all of the steps they would take to solve the problem and to write at least one sentence explaining their work. The results were that only 22 out of 59 students used a correct method to solve the problem. Twenty-one others solved by again "canceling" the log notation and solving the remaining linear equation, and 16 could not do the problem at all. This problem was explored in more detail in Phase II.

Phase II

In the interviews, the students were given three tasks and asked to "think out loud" and explain how and why they solved each problem. The subjects, Lynn and Anna, were guided to make some connections among their work on all of the problems. To help distinguish between the students' meaningful knowledge of facts and rote-learned facts, analysis included comparisons of each interviewee's solutions to the three tasks as well as some comparison between the two students (Gray & Tall, 1994).

For task one, students were asked to solve for *x* given the equation

$$\log_5(x) + \log_5(x+4) = 1. \qquad (1)$$

Both Anna and Lynn used the same incorrect methods to solve Equation 1 as they had used on a similar problem on their classroom test. For Lynn, the problem brought to mind a procedure, which was itself incorrect and provided no indication of a concept understanding of logarithms. Her method was simply to eliminate the logarithmic notation from the problem entirely and then solve the remaining linear equation. The following transcript describes her thinking:

Lynn: First set these [expressions of *x*] equal since logs are the same, so I'll get 2*x* plus 4 equals 1. Subtract the 4, divide by 2.

Interviewer: What about this problem allowed you to do that?

L: Well, um, the logs are the same, the fives, so when they're the same you can assume that they're equal so you can solve it like that.

I: Okay so what happens to log?

L: I guess they cancel out.

I: What does that mean to you?

L: They're both, we don't have to include them. They both cancel out and that gives you that there (pointing to her result). They're equal or, (pause) I guess it's like when you have a negative 1, or a 1 and a negative 1. That would give you zero, so that's how they cancel out. You don't include them.

Lynn interpreted the presence of two log functions with the same bases as things that negated one another, although her comparison to adding 1 and –1 is surprising and her reasoning here is unclear. When asked if in her first line of work that showed $x + x + 4 = 1$, she thought the x's "cancelled out" because they were the same, she said no, that that was just a property of logarithms. It is apparent that she not only lacked a conceptual understanding of log functions, but also a correct procedural understanding for this problem. Lynn did not see logarithms as objects to be manipulated, in fact, she saw no need for the logarithmic notation to remain with the problem, even when checking her answer of x= -3/2 in the original problem. The notation did not evoke any anticipation of a need to change the function to exponential form. It is most likely she was simply remembering from class work that the log notation "disappeared" on these types of problems, and was applying a method that she had identified as necessary for making this happen. Lynn's' solution method is an example of how, in the absence of a proceptual understanding of a logarithmic expression as both an indication of the exponentiation process and an object to be manipulated to be used in a procedure, students will make up their own rules in math to make their solution match what they remember from previous examples. This supports Gray and Tall's (1994) idea that less-able students are not learning correct techniques more slowly, but are instead developing their own techniques. In fact at the end of the interview, after the researcher helped Lynn recall properties of logs so that she could produce the correct solution to this problem, she admitted that she remembered doing the problem that way, but that her method "seemed so much easier." She made no acknowledgement of her method violating any laws of mathematics.

Anna, on the other hand, initially recognized the logarithms in Equation 1 as objects that could be manipulated and that needed to be converted into a new object before processes could be applied. However, she used the properties of logs incorrectly and dropped the logarithmic notation from the problem, producing $x(x+4) = 1$. She, like Lynn, did not possess any real understanding of the concept of logarithms or any anticipation for using exponentiation in the solving process. She simply solved the resulting quadratic equation and obtained two irrational solutions, which she did not check back in the original problem. Her work is described below:

I: So what happened to log? Why is log in this first line but not the second? Anna: They cancel out.

I: Can you explain?

A: I don't know. I think that's how I remember it is they cancel out.

I: What does canceling out mean to you?

A: They have a, I don't know if complementary is the right word, but another one was there...so that positive or negative or something like that. Two of something.

I: When you look here [pointing to x (x+1)] you think multiplication. When you look here [pointing to log5(x+4)] do you think multiplication?

A: No…this I see as something with like an exponent like 5 to the something equals x +4. Anna was relying primarily on memorized facts, justifying her thoughts by saying what she recalled from class. She also exhibited algebraic misconceptions about the concept of canceling.

Anna's last statement in the transcript is interesting because it shows that when given a single log expression, the notation suggested a process of exponentiation. Both Anna and Lynn further demonstrated procedural understanding on the other tasks. For example, students were shown twelve index cards containing logarithmic expressions or equations for task two, and were asked to talk about what came to their minds when they saw each card. On most of the index cards involving a single equation, both students consistently converted the log equations to exponential form and then solved the resulting equation for *x*. Similarly, both students were able to use the correct procedure to solve the problem

$$\log_7(2x+1) = 2 \qquad (2)$$

for task three. For these types of problems, it might be tempting to suggest that the students had some conceptual, or even proceptual understanding of the logarithms. However, their approaches to the first problem leads this researcher to believe that their abilities to solve problems like Equation 2 were based on rote-learned fact, and not an example of a meaningful knowledge of logarithms that could be used to derive new facts or solutions (Gray & Tall, 1994; Tall et al., 2000). The introduction of a second log term in the problem completely changed the students approach to the problem. This may be due to the students' fixation on the process aspect of these problems. According to Tall et al, "knowing a specific procedure allows the individual to do a specific computation" (p. 8). In other words, viewing Equation 2 as only a process of converting to exponential form limits students' ability to see the expression as an object available for further manipulation in a more complicated equation like Equation 1 (Gray & Tall, 1994; Kinzel, 1999). These students lacked the proceptual understanding needed to be able to interpret the log symbolism in a flexible way, which is essential for successful mathematical thinking (Gray & Tall, 1994; Tall et al, 2000). These results are not unique to the two interviewed students as indicated by the results on the questionnaires.

A few other results from task two are worth mentioning here. When considering the index cards, both students struggled with the expression

$$\log_4(-16). \qquad (3)$$

Both girls converted this to exponential form, and then proceeded to guess and check answers on their calculator. They thought that the answer should either be a negative number or a fraction, and even though they could not come up with a result, they both suggested that there probably would be one if they kept looking. The fact that they believed that four raised to a power could produce a negative result is an indication of a major lack in understanding of both a process and concept of exponentiation. According to Weber (2000a, 2000b), a process understanding of exponents is the starting ground for building an understanding of exponential and logarithmic functions, which could help explain the difficulty students had solving logarithmic equations.

A problem with natural log was witnessed again in the interviews where, when shown an index card with the expression log (10), both of the interviewed students were able to determine that the answer was 1 because 10 raised to the 1-power equals 10. When shown ln(*e*), they also answered that it was 1, but neither student could provide more explanation than it was a known fact from class that natural log and *e* "cancel each other out." They were relying on rote-learned facts because the notation did not call to mind any other available tools for solving this problem or verifying the known result.

It is also interesting to note, that when shown the index card log(*xy*), Lynn recognized this symbol as an object that could be decomposed by the laws of logarithms to log(*x*) + log(*y*), but that the expanded form in Equation 1 in the first task problem did not suggest the use of this law. This indicates that Lynn's understanding of the concepts of expanding and condensing log functions was based primarily on memorized facts, and her inability to recall these facts caused her to solve the problem incorrectly.

Conclusion

The results of this study indicate that the students did not, in general, have a proceptual understanding of logarithms. The presentation of single logarithmic forms evoked the procedural response of rewriting the problem in exponential form in almost all cases. However, the addition of a second log form to the equation no longer prompted students to anticipate a change to exponential form; their learned procedure no longer fit the given form of the equation. In fact, they failed to demonstrate any process or object understanding of logarithms, and instead invented their own solution methods for "getting rid of" the logarithmic notation and finding x. It does seem, as Gray and Tall (1994) suggest, that less successful students are not necessarily slower at learning correct methods and in need of more time to understand concepts, but are simply doing different mathematics than the more successful mathematical thinkers. Furthermore, it is important to note that the students examined in this study showed no indication of moving away from their different methods or recognizing their faults. This is exemplified by the fact that both Anna and Lynn had already been tested on logarithms and had received their tests back with corrections, yet they both answered Equation 1 using the exact same incorrect method they had employed on their tests. If students are not held accountable for using material learned in math classes except on a single test, or perhaps twice if the material is revisited on the final exam, then there is little incentive for them to learn from their mistakes.

The major problem that students seem to have with logarithms is making meaningful connections to the name logarithm and the notation used to represent it. For example they may be able to relate to the notation $f(x) = x^2 + 3$, and give it a literal interpretation such as f is the function that takes a number for x, squares it, and adds three, but logarithmic notation is much more ambiguous; it does not "tell" students what to do. Without a deep understanding of its function and use, students must rely on memorization, which usually proves unsuccessful in the long run. For proceptual thinking to exist, the concepts of logarithms and exponentials should be as closely linked as addition and subtraction, where exponentiation is understood simply as a flexible reorganization of logarithmic facts (Gray & Tall, 1994). Teachers need to find ways of doing more than just teaching the processes and properties involved with logarithmic functions and instead make logarithms objects to which students can relate. With a meaningful understanding, students may be able to think about functions such as logarithms as both processes and conceptual objects and become more successful advanced mathematical thinkers.

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