ADDRESSING THE REVERSAL ERROR WITH A VISUALIZATION TOOL

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In this study, we examine engineering students' engagement with the common Student-Professor type problems using a web-based tool designed to develop a conceptual understanding of formulating valid mathematical models. We analyze one student's thinking on five tasks with this tool. Findings suggest that interacting dynamically with verbal, visual, symbolic, and numeric representational spaces of such problems allowed the student to better understand the relationship between coefficients and variables.

Keywords: Algebra and Algebraic Thinking, Modeling, Technology

Background

Consider the well-known problem: Write an equation to represent the following statement: 'There are six times as many students as professors at this university.' Many studies have examined the "student-professor problem" and issues surrounding it (see e.g., Clement, 1982; Cohen & Kanim, 2005). In spite of its seeming simplicity, this problem and its variations (henceforth called SP-type problems) have caused difficulty for students at all levels. The most common error is the *reversal error*; writing $P = 6S$ instead of $S = 6P$. This error has been studied repeatedly, yet its complexity continues to intrigue researchers (Cohen & Kanim, 2005). We have encountered this error while working with college engineering students. In this paper, we focus on the research question: in what ways can an interactive visualization tool help college students analyze and makes sense an SP-type problem? Because of the long history of student difficulties with SP-type problems, we believe this is a useful place to focus our attention.

Many researchers have attributed difficulty with SP-type problems to a "direct-translation" approach (Clement, 1982; Fisher, Borchert, & Bassok, 2010). With this strategy, one makes the sequence of algebraic symbols match the sequence of objects in a word problem. For example, given "There are six times as many students as professors," the reversal error is made by translating directly to 6 times *S* equals *P*. MacGregor and Stacey (1993), however, found that when phrasing questions so that a direct translation produces a correct response, students still tended to make the error. They attribute difficulties to an underlying cognitive model of mathematical relationships in which models are based on comparison rather than equality. Others have identified this phenomenon as using a "static comparison" strategy (Clement, 1982; Cohen & Kanim, 2005), where "$6S$ " and "P " are treated as six students and one professor respectively (instead of as the number of students and professors). The equal sign is seen here as representing correspondence rather than equality (Cohen & Kanim, 2005; Palm, 2008).

Two distinct behaviors have been seen in students who successfully solve simple word problems: An *operative approach* involves the use of a hypothetical operation that produces an equivalence relation (Palm, 2008). Fisher et al. (2010) found that writing a non-standard relationship by developing such equivalence relationships decreased the appearance of the

Kenney, R., An, T., Kim, S.H., Yi, J.S., Uhan, N. (In press). Addressing the reversal error with a visualization tool. Accepted for the *Thirty-first Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA)*, Chicago, IL.

reversal error significantly. The second pattern is what Clement (1982) called the *substitution pattern*, where students substitute numbers into the equation's variables and then engage in the operative approach.. In this study, we encourage students to engage in the patterns suggested above through the use of visualizations of the problem space that we create though our Room-Metaphor and Test-Case approaches (described below) in a computer environment. Yazdani (2008) also found that asking students to draw a picture, figure, table or graph greatly improved students ability to grasp problems on which reversal errors had previously been made and found that visualizations helped students detect errors and see limitations in previously used strategies. In this study, we focus on a visualization of the mathematical expressions themselves in an effort to further enhance students' abilities to solve word problems without the reversal error.

Framework

Zazkis, Dubinsky, and Dautermann's (1996) Visualization/Analysis (V/A) model suggests that an integration and interplay of visual and analytic thinking is often needed for understanding mathematical concepts. The V/A model begins with an act of visualization, which can be the actual drawing of a picture or the expression of a mental image. This is followed by an act of analysis in which the person reasons about what was visualized. Then follows a second visualization step, enriched by the analysis, which then leads to a second act of analysis, etc. (Stylianou, 2002). As the learner develops a clearer understanding of a mathematical problem, the iterations increase in sophistication and enhance conceptual understanding (Zazkis et al., 1996; Stylianou, 2002). Stylianou (2002) expanded the framework to identify particular "analysis steps" which we have elected to use as codes (Table 1) in our data analysis to address ways in which visualization helps a learner understand SP-type problems.

Table 1: Data Codes

Code	Definition
IAC	Collect new info from a visualization to understand implications of prior actions;
MEI	Use new info gained to further explore the problem; Use patterns to refine visualization
SNG	Create a visualization to better understand the problem or next steps in problem solving
MSP	Pose questions (to one's self) to check understanding and verify correct procedures.

Methods

We interviewed engineering students as they solved SP-problems to identify possible designs for our tool, resulting in our development of two interventions to use in tandem. First, we developed an interface to prompt the use of test cases and provide instant feedback when they were not compatible with constructed equations. This strategy was included with the intention of encouraging students to adopt a substitution pattern of thinking (Clement, 1982). We also developed a second, pictorial approach called the "room metaphor" approach, which involves showing a variable as a room that holds some number of boxes. This room metaphor addresses the static comparison approach mentioned above and can help students build an understanding that a variable is not just an object, but also a "container" of objects. When an equation is entered and test cases are specified and "hovered" over with the mouse, black boxes appear in the "rooms" to show a variable has taken on a value. If values correctly satisfy the equation, black boxes will fill the blocks perfectly. For a mismatch between equation and test cases (e.g., $P = 6S$, $P = 1$ and $S = 6$), empty spaces appear as a visual cue to indicate an error (see Figure 1).

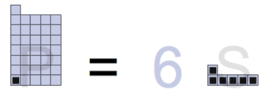

Figure 1: Feedback With Incorrect Equation And Correct Test-Case Values

To test the tool, we created five SP-type problems (see Table 2).

Table 2: Tasks Used in the Study

P1. A shoe store has four times as many pairs of leather boots as suede boots. Write an equation to represent this relationship.

P2. The gravity on the earth is six times greater than on the moon. Write an equation about this relationship.

P3. A country exports twelve times as much oil as coal. Write an equation for the relationship between the amount of oil and coal.

P4. The Yi Company produces chairs that have a seat piece and four leg pieces. Write an equation representing the relationship between the number of seats and legs.

P5. A landscape designer is designing a garden. He plans to insert six cherry trees for every ten holly trees. Write an equation using *C* and *H* to represent the above relationship.

Findings

We share our analysis of one case study, Nicole, a junior Industrial Engineering major. In P1 and P2 Nicole was able to create a correct model with little trouble. In both cases, there was evidence that she took time to think about the problem before coming up with the model. For example, she told the interviewer, "I had to figure out which side the six and one need to be on." On problem 3, however, Nicole encountered difficulty. Table 3, shows segmented transcript lines of this problem with accompanying data codes.

Table 3: Transcript Analysis of P3

Line	Transcript	Code
1	**N:** A country exports 12 times as much oil as coal. So coal equals oil	SNG
2	And if you have one coal, then you have 12 oil.(types $C = 12O$)	IAC
3	If you have 12 coal, you have one oil (Types C=12, O=1 in test cases).	MEI
4	(Pause while she studies problem and hovers over test cases).	SNG
5	If you have...you know you need more…. Twelve times as much oil as coal. So there is more oil than there is coal, so my equation is flipped?	IAC
6	(pauses) Yes! Ok, my equation is flipped.	MSP
	***I:** Do you feel the picture helped you figure out there was something wrong here?*	
7	**N:** In this case it did help me since reading the problem, you need more oil than coal so when you see the boxes with variables, you know I wrote the equation wrong, because I was like, why do I have so many coals? (refers to picture).	MSP
8	I need more oils than I have coals.	IAC
	***I:** If you had put the test case in first in this one instead of the equation, do you think it could have helped you avoid the mistake?*	
9	**N:** The test case might have helped me think about the equation, but it is a little confusing that the 12 goes...like in the equation it's 12 *C*, but when you do the test cases it's 12 under the *O* column.	MEI

In line 2, Nicole seems to have used a static comparison strategy where *C* stands for "coal," not amount of coal. She states that if you have 1 coal, you have 12 oil, but then in line three she uses the equation she has created rather than her previous thinking to put in the test cases *C*=12 and *O*=1. In line 4, something perturbs Nicole's thinking as she hovers over her test cases in the tool. She uses the visualization produced to understand the implications of her placement of the 12. In lines 7 and 8, she explains that the fact that there were more boxes attached to coal when she expected more under oil changed her thinking about the problem. Even though her test case and equation matched, the room metaphor helped her recognize an error in her thinking. Without this perturbation, Nicole admitted that she "probably would have just gone with her first answer."

In response to the interviewer's question in line 8, she identifies that it is counterintuitive to know that the test case value of *O* is 12, but that in the equation the 12 is attached to the *C*. This pattern that she now notices and reflects on in lines 9 represents a turning point in Nicole's thinking about these problems, making the MEI analysis a key step in her thinking.

In P4 and P5, Nicole repeatedly referred back to what she learned from her mistake in P3. In P4, she explains, "Well, for every one seat you need four legs. So since I messed this up in the last equation, I know that 4 goes with the *S*, and that equals *L*." From her discussion, it seems reasonable to assume that Nicole is not just creating the correct equation by blindly following a new procedure she developed in her work on P3, but can explain the relationship between S and L as an equivalence relationship rather than as a ratio or correspondence.

Conclusions

We feel the dynamic connection of numeric, verbal, and symbolic representations helped Nicole interact with the problem more efficiently. She used visualizations to collect useful information, allowing her to view the problem from different perspectives and further understand the need for a balanced relationship in the model. In general, when working with simple linear relationships, we need to help students move away from static comparison and direct translation habits and understand the equal sign as representing a balanced relationship. We feel this tool is one useful way of helping students reorganize their thinking on these problems.

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