Review Sheet for Mid 1
Math 266
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Disclaimer: This sheet is neither claimed to be complete nor indicative and may contain typos.

1. Types of equations and techniques

1.1. First order Linear Equations.

(1) \( \frac{dy}{dt} + p(t)y = g(t) \)

Solution:

(2) \( \mu(t) := \exp\left(\int p(t) dt\right) \)

(3) \( y(t) = \frac{1}{\mu(t)} \left( \int \mu(t) g(t) dt + c \right) \)

The initial condition determines \( c \). E.g. \( y(0) = y_0 \).

1.2. Separable Equations.

(4) \( \frac{dy}{dx} = \frac{g(x)}{h(y)} \)

Solution:

(5) \( h(y)dy = g(x)dx \)

(6) \( \int h(y)dy = \int g(x)dx + c \)

Get implicit solution: \( \int h(y)dy - \int g(x)dx = c \).

1.3. Exact Equations.

(7) \( M(x,y)dx + N(x,y)dy = 0 \)

Criterion:

(8) \( M_y = N_x \)

Conditions: \( M, N, M_y, N_x \) continuous on simply connected region (e.g. rectangle).

Solution:

(9) \( Q(x, y) = \int_{x_0}^{x} M(s,y)ds \), defines \( Q \) and solves \( Q_x = M \)

(10) \( \psi(x, y) = Q(x, y) + h(y) \), defines \( \psi \) in terms of \( Q \) and \( h \)

(11) \( \psi_y = Q_y + h'(y) = N_x \), gives differential eq. for \( h \)

(12) \( h(y) = \int Q_y - N_x dy \), defines \( h \)

(13) \( \psi = c \), gives the answer

Get implicit solution.
1.4. **Integrating factors.** Multiply equation (7) by $\mu(x, y)$

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

**Criterion:**

$$(\mu M)_y = (\mu N)_x$$

**Special case**

$\mu(x, y) = \mu(x)$ depends only on $x$. Test: $(M_y - N_x)/N$ only depends on $x$.

**Solution:**

1. $$\frac{d\mu}{dx} = \frac{M_y - N_x}{N} \mu$$

   gives ODE for $\mu$. Solve!

2. With $\tilde{M}(x, y) := \mu(x)M(x, y)$, $\tilde{N}(x, y) = \mu(x)N(x, y)$, get new exact equation:

$$\tilde{M}(x, y)dx + \tilde{N}(x, y)dy = 0$$

Use algorithm above for exact equations.

2. **Qualitative behavior of autonomous equations**

$$\frac{dy}{dx} = f(y)$$

2.1. **Phase line.**

1. Find constant/equilibrium solutions: $y : f(y) = 0$.
2. Draw vertical line and mark off constant solutions.
3. In each interval between constant solutions determine the sign of $f(y)$ and draw arrow (up if $f(y) > 0$, down if $f(y) < 0$). This means that the graph of $y$ is going up or down for values of $y$ in the respective interval.
4. Constant solutions are:
   - stable if both arrows point towards
   - unstable if both arrows point away
   - semi-stable if one arrow points toward and one away from the particular point of constant solution

2.2. **Second order behavior of solutions.** Need to consider $f(y)$ and $f'(y)$ for behavior of graph of $y(x)$

$$f(y)f'(y) > 0 \quad \text{convex}$$
$$f(y)f'(y) < 0 \quad \text{concave}$$
$$f(y)f'(y) = 0 \quad \text{may be an inflection point.}$$

Notice $f(y) = 0$ means constant (equilibrium) solution so $f'(y) = 0$ are the interesting points of inflection.
3. Setting up equations

3.1. Growth. Determine population \( P \) as a function of time given \( r \) rate of reproduction and \( k \) “kill” rate.

\[
\frac{dP}{dt} = rP - k
\]  

Special case: Rate \( r \) given indirectly: e.g. Population increases by factor \( s \) after each time interval if system is unperturbed.

Solution: unperturbed equation \( \frac{dP}{dt} = rP \) has solution \( P(t) = P_0 e^{rt} \) so \( s = P(t + 1)/P(t) = e^r \) and hence \( r = \ln(s) \).

3.2. Flow. Amount of substance \( Q \) in a liquid system with liquid inflow rate \( r_{in} \) and outflow rate \( r_{out} \). Let \( V(t) \) be the volume of system and \( C \) the inflow concentration.

\[
\frac{dQ}{dt} = \text{substance in rate} - \text{substance out rate}
\]

\[
\frac{dQ}{dt} = r_{in}C - \frac{r_{out}Q}{V(t)}
\]

Special cases:
1. \( V(t) = V_0 \) fixed \( r_{in} = r_{out} \)
2. \( V_0 \) initial volume: \( V(t) = V_0 + (r_{in} - r_{out})t \)

4. Theorems on existence and continuity

4.1. Linear first order equations.

\[
\frac{dy}{dt} + p(t)y = g(t)
\]

If \( p(t) \) and \( g(t) \) are continuous on \( I \) then solutions exist, are unique and continuous on \( I \).

Discontinuous or singular points can only appear at discontinuities or singular points of \( p(t) \) and \( g(t) \). You do not have to solve the equation to determine the maximum intervals of definition and continuity of the solution given an initial time \( t_0 \).

4.2. Non-linear first order.

\[
\frac{dy}{dt} = f(t, y)
\]

If \( f \) and \( \partial f/\partial y \) are continuous on rectangle \( R \) then solutions with initial conditions \( (t_0, y_0) \) exist and are unique in an interval \( J \), s.t. \( t_0 \in J \) and all points \( (t, y_0) \in R \) for \( t \in J \).

Notice \( J \) may not be maximal, i.e. it does not have to include all points \( t \) with \( (t, y_0) \in R \). You have to solve the equation to know the maximal domains of continuity and definition of the solution given an initial point \( (t_0, y_0) \) often given as \( y(t_0) = y_0 \). The important data is \( t_0 \).

At points of discontinuity of \( \partial f/\partial y \) with \( f \) continuous, existence is guaranteed, but more that one solution can exist.