The geometry of Moduli Spaces of pointed curves, the tensor product in the theory of Frobenius manifolds and the explicit Künneth formula in Quantum Cohomology

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Introduction

In the last decade, a new branch of mathematics grew out of the interaction of theoretical physics, namely string theory, and mathematics. In an attempt to find a mathematically rigorous formulation of the fundamental paper of Witten [W] and thus to explain some of the striking numerical predictions made for instance by the Austin physics group of Candelas [COGP] using the so called Mirror Symmetry emerging from string theory, the theory of Gromov–Witten invariants and several mathematical versions of Mirror Symmetry (MS) were created.

In the physical formulation, Mirror Symmetry basically states that two string theories with different target spaces (the so–called A–model and B–model) lead to the same conformal field theory on the worldsheet. This has several consequences, e.g. in an imprecise formulation that for each Calabi–Yau quintic threefold with Hodge numbers $h^{1,1} = a$ and $h^{1,2} = b$ there exists a dual Calabi–Yau with the mirrored Hodge numbers $h^{1,1} = b$ and $h^{1,2} = a$. Much evidence of this nature was presented in the above mentioned collaboration. In the guise of Hodge number duality the mirror symmetry has been rigorously formulated and was proven to exist in the case of Calabi–Yau$s$ which can be realized inside toric spaces by Batyrev [Ba].

A more sophisticated version of MS relates two generating functions: on the A–side a generating function for the "number of curves" and a hyper–geometric series which satisfies certain Picard–Fuchs equations on the B–side. The term "number of curves" has been made precise by the theory of Gromov–Witten invariants [G, W] which were, in the algebraic context, first introduced in [KM] on an axiomatic basis and later proven to exist [BM, Be1] by constructing a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{virt}$ of the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(V,\beta)$ introduced by Kontsevich. The above mentioned space parameterizes stable maps of n–pointed curves of genus g into a given projective smooth manifold V and can be viewed as an extension of the classical Delinge Mumford Knudsen spaces of stable curves of genus g with n marked points $\overline{M}_{g,n}$. In the symplectic setting similar results were proven by Ruan and Tian [RT] and Li and Tian [LT]. The MS in this formulation has been proved by Givental for the case of projective complete intersections [Gi] by using equivariant cohomology in order to enhance the approach via torus action suggested by [K2]. A review of these constructions can be found in [V].

The most far reaching statement of MS has been given by Kontsevich in [K3] where the so–called homological MS is thought of as relating two triangulated categories. It is conjectured that the bounded derived category of coherent sheaves of one manifold is equivalent to the bounded derived category obtained from the Fukaya category of a dual manifold.

In the present exposition we concentrate on the A–side of this general picture and deal with the theory of Gromov–Witten invariants. The Gromov–Witten invariants can be regarded as a precise definition of what is meant by "the number of curves on a variety V with certain incidence conditions". These numbers coincide under certain conditions with the actual number of these curves. Motivated by physics, one collects "all" of these solutions to the corresponding enumerative problems in a generating series, the Gromov–Witten–potential. The fact that the GW–numbers can be regarded as the correlation functions of a topological field theory or, more precisely, as the codimension zero correlation functions of a Cohomological Field Theory (CohFT), corresponds to a mathematical property of the GW–numbers which leads to a differential equation for the generating series, the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) or associativity equations. In the case of Fano varieties, these equations can be used to provide recursion relations for the GW–numbers, thus reducing "all" problems to certain basic ones (see [KM]).

So far, only the GW–numbers which represent the codimension zero correlation functions of a CohFT have been discussed. The GW–invariants, however, also furnish all higher codimension correlation functions. These correlation functions are defined as maps from $H^*(V)^{\otimes n}$ to the cohomology of the moduli space of stable n–pointed curves of genus g, $H^*(\overline{M}_{g,n})$. In [KM, KMK] it was shown that in genus zero all these maps can be reconstructed using the GW–numbers. For the proof of this statement and the explicit reconstruction of the GW–classes from the GW–numbers in genus zero, one needs to study the intersection theory on the moduli space of stable n–pointed genus zero curves $\overline{M}_{0n}$. In particular, a basis of $H^*(\overline{M}_{0n})$ and its intersection matrix needed for the explicit reconstruction is presented in this exposition.

In genus zero the theory appears in various alternative formulations. Two of these formulations were already mentioned: the CohFT approach and the approach using just the codimension zero correlation functions which in a general framework are an instance of Abstract Correlation Functions (ACFs). Thanks to the associativity equations one can define a commutative and associative algebra with the help of the third derivatives of the GW–potential, the so–called quantum cohomology ring. Starting from the data of the linear super–space $H^*(V)$ together with the GW–potential Φ regarded as yielding the quantum multiplication on the tangent space to $H^*(V)$, one is lead to the notion of a formal Frobenius manifold. Following this starting point or more generally analyzing topological field theories together with their natural moduli spaces, Dubrovin founded the theory of Frobenius manifolds [D] which forms a third vantage point for the genus zero theory. There is yet another formulation of CohFT avoiding the explicit mentioning of the moduli spaces via the structures of $Comm_\infty$–algebras [Ge1] which is up to a dualization equivalent to the formulation in terms of ACFs.

In the realm of CohFT, there is a natural operation of forming the tensor product which —translated in terms of quantum cohomology— yields the Künneth formula (cf. [KMK] and Chapter 3). Turning to the language of operads [Ge1, GK], this reflects the fact that the $H_*(\overline{M}_{0n})$ operad is an operad of coalgebras and not just of linear spaces. In the $Comm_\infty$ picture, this operation is not naturally seen and quite an amount of work is needed to translate this operation into this setting.

In the present exposition, we perform an in–depth analysis of the operation of forming the tensor product in the various formulations of the genus zero theory (CohFT, ACFs and Frobenius manifolds) with the help of intersection theory on the moduli space of stable n–pointed genus zero curves $\overline{M}_{0n}$ and study the interrelation between the different presentations. In this way, we obtain a presentation of the operation of forming the tensor product in all presently known facets of the genus zero theory.

In two additional sections, we review the results of the collaboration [KMZ] concerning higher Weil–Petersson volumes and one–dimensional CohFT, putting emphasis on our part of the proofs.

Pertaining to the theory of Frobenius manifolds, we provide the explicit formula for the tensor product of formal Frobenius manifolds and derive the tensor product in the enlarged category of formal Frobenius manifolds with an Euler field and a flat identity. Furthermore, we introduce the notion of forming the tensor product germs of pointed Frobenius manifolds and give a complete description of the dependence of this operation on the base–point. In the special case of semi–simple Frobenius manifolds, this leads to a formula for the special initial conditions which basically determine the structure of these manifolds.

In the setting of quantum cohomology, we derive the explicit Künneth formula for quantum cohomology.

The exposition is organized as follows:

In Chapter I, we begin by recalling the necessary basic definitions concerning the moduli spaces $\overline{M}_{g,n}$. After providing the intersection formula for two strata classes of complementary dimension, we set out to prove a formula for the intersection of an arbitrary number of strata classes. To this end, we introduce the new notions of trees with multiplicity and multiplicity orientations. Using these definitions, we prove the mentioned formula. In the following section, we present a basis for the cohomology of $\overline{M}_{0n}$ and use the results of the previous section to derive its intersection form. We then provide a formula for the inverse intersection matrix, which, together with the basis, supplies a representative of the diagonal class $\Delta_{\overline{M}_{0n}}$ of $\overline{M}_{0n} \times \overline{M}_{0n}$ and give examples of the intersection matrices for small values of n.

The last section of Chapter I deals with a second aspect of the geometry of the spaces $\overline{M}_{g,n}$, the study of a generalization of Weil–Petersson volumes which has been the subject of the collaboration [KMZ]. Having defined these higher analogs, we collect them into a generating series and derive a recursion relation for this series in genus zero again using explicit formulas in the cohomology ring of $\overline{M}_{0n}$. We continue this section by quoting the further results of [KMZ] which are used in the next chapter.

The second chapter, Chapter II, is devoted to the study of the tensor product in the theory of Frobenius manifolds. We begin by reviewing the basic notions of Frobenius manifolds, formal Frobenius manifolds, semi–simple Frobenius manifolds, the additional structures of an Euler field and a (flat) identity and introduce the notion of pointed Frobenius manifolds. Then, we turn to the tensor product for each of these types of Frobenius manifolds.

In the case of formal Frobenius manifolds, the tensor product is defined via the connection to ACFs or $Comm_\infty$–algebras. Since the tensor product is only naturally defined on CohFT, one has to first reconstruct the relevant CohFTs from the ACFs and only afterwards can one reduce to the ACFs of the tensor product CohFT. The theoretic framework for this construction is provided by passing to an operadic version of ACFs which again takes the geometry of $\overline{M}_{0n}$ into account. In this setting, we use the results of Chapter I to give an explicit formula for the potential of the tensor product.

In the following section, we make a digression to the rank or dimension one CohFTs and the tensor product in this subset of CohFT. The theory of these CohFTs is equivalent to the study of the higher Weil–Petersson volumes of the previous chapter. Translating the results into this language yields a complete description of the tensor product for the potential as well as explicit formulas which we quote from [KMZ].

We continue by translating the conditions for an Euler field and a flat identity into relations among the operadic correlation functions. We then present an Euler field and a flat identity for the tensor product of two Frobenius manifolds, if the factors also carry these additional structure. The proof that the proposed candidate indeed satisfies the definition of an Euler field is non–trivial and, besides involving the properties of the Euler field and the identity in the setting of operadic ACFs, it additionally relies on properties of the diagonal class of $\overline{M}_{0n} \times \overline{M}_{0n}$ under the push–forward and pull–back with respect to the forgetful morphisms.

Using the formulation for formal Frobenius manifolds, we transfer the operation of forming the tensor product to germs of pointed Frobenius manifolds under certain convergence conditions. We then study the dependence on the choice of the base–point and prove a theorem that basically states that up to the mentioned convergence conditions the tensor product for germs of Frobenius manifolds is independent of the choice of base–points up to unique isomorphism.

In the case of split semi–simple Frobenius manifolds, these convergence conditions are automatic and the above considerations provide a base–point–free formulation of the tensor product in this class. Moreover, according to [M3] a split semi–simple Frobenius manifold is already determined by the initial data for a particular differential equation. Using our theorem about the tensor product for the Euler fields, we derive the initial data for the tensor product of two such manifolds in terms of the initial data of the factors.

In the last chapter, Chapter III, we focus on the aspect of quantum cohomology and apply the previously established results to this situation. In particular, as mentioned above, the explicit formula for the tensor product of two formal

Frobenius manifolds turns into the explicit Künneth formula for quantum cohomology which expresses the quantum cohomology of a product variety $V \times W$ in terms of the quantum cohomology of the factors V and W.

We end this exposition by giving several examples. Namely, we provide the special initial conditions for a product of projective spaces and calculate the GW–potential of a product of two and three three–dimensional Calabi–Yau manifolds.

CHAPTER 1

Moduli spaces of stable curves

In this chapter, we deal with the geometry of the moduli spaces of stable curves $\overline{M}_{g,n}$. After recalling some of the basic definitions, we present our results concerning the intersection theory on $\overline{M}_{0n}$ and on the so–called higher Weil–Petersson volumes. We will, in particular, provide formulas for the intersection of strata classes and monomials of boundary divisors in $\overline{M}_{0n}$. Furthermore, we present a basis for $H^*(\overline{M}_{0n})$ and its intersection form as well as the inverse intersection form. In an additional section containing the results of [KMZ], we will reproduce the proof of a recursion relation for the generating function of higher Weil–Petersson volumes.

1. The moduli stacks $\overline{M}_{g,n}$

The moduli spaces of smooth Riemann surfaces of a given genus were classically already studied by Riemann himself. In a more precise framework, the respective moduli stacks to the coarse moduli problem which are denoted by M_g were introduced (see e.g. [MF]). Deligne and Mumford [DM] developed a compactification scheme for these spaces yielding the stacks $\overline{M}_g$. This compactification basically adds the locus of degenerate curves with simple double points as boundary divisors. The latter author also initiated the algebro–geometric study of the Chow ring of these spaces. In [Mu], he introduced a series of classes which were consequently named after him and gave a complete description of the Chow ring in the case $g = 2$. For higher genus explicit descriptions for $g = 3$ and some results for $g = 4$ were obtained by Faber [Fa].

Often one additionally includes the additional structure of marked points into the moduli problem thus considering curves of genus g with n marked points. Following Deligne and Mumford, Knudsen [Kn] developed a compactification $\overline{M}_{g,n}$ for the appropriate moduli stacks $M_{g,n}$.

An excellent detailed account of what is currently known about these moduli spaces and and their mapping class groups can be found in the recent article [HL]. Thus, we will only repeat briefly the notions which are essential to our work.

1.1. Pointed curves and their graphs. In this subsection, we cite the main definitions of the general geometric objects under investigation in the formulation of [M2].

1.1.1. Definition. A prestable curve over a scheme T is a flat proper morphism $\pi : C \to T$ whose geometric fibers are reduced one–dimensional schemes with at most ordinary double points as singularities. Its genus is a locally constant function on $T : g(t) := \dim(H^1(C_t, \mathcal{O}))$.

1.1.2. Definition. Let S be a finite set. An S–pointed (equivalently, S–labeled) prestable curve over T is a family $(C, \pi, x_i | i \in S)$ where $\pi : C \to T$ is a prestable curve and the x_i are sections such that for any geometric point t of T we have $x_i(t) \neq x_j(t)$ for $i \neq j$ and the x_i are smooth on C_t. The points $x_i(t), i \in S$ and the singular points of C_t are called special points.

Such an irreducible curve is called stable, if $2g - 2 + |S| > 0$ and every non–singular genus zero component of any C_t contains at least three special points. A general prestable pointed curve is called stable, if all its connected components are stable.

1.1.3. Definition. A finite graph τ is a quadruple $(F_\tau, V_\tau, \partial_\tau, j_\tau)$ of a (finite) set of (of flags) F_τ, a (finite) set (of vertices) V_τ, the boundary map $\partial_\tau : F_\tau \to V_\tau$, and an involution $j_\tau F_\tau \to F_\tau, j_\tau^2 = j_\tau$.

An isomorphism $\tau \to \sigma$ consists of two bijections $F_\tau \to F_\sigma, V_\tau \to V_\sigma$, compatible with ∂ and j.

The two–element orbits of j_τ form the set of edges E_τ and the one–element orbits form the set of tails T_τ.

1.1.4. Geometric realization. Given a graph τ define for each vertex v the set $F_\tau(v) = \partial_\tau^{-1}(v)$ and consider the topological space called "the star of v" consisting of $|v| := |F_\tau(v)|$ semiintervals having one common boundary point. These semiintervals are labeled by their respective flags. Then glue these stars according to j_τ yielding edges and tails.

A graph τ is called connected (respectively simply connected), if its geometric realization $||\tau||$ has this property.

1.1.5. Definition. A modular graph is a graph τ together with a map $g : V_\tau \to \mathbb{Z}_{\geq 0}, v \to g_v$. An isomorphism of two modular graphs is an isomorphism of the underlying graphs, preserving the g–labels of the vertices.

A modular graph (τ, g) is called stable, if $|v| \geq 3$ for all v with $g_v = 0$ and $|v| \geq 1$ for all v with $g_v = 1$.

1.1.6. Definition. The dual modular graph (τ, g) of a prestable S–pointed curve $(C, \pi, x_i | i \in S)$ over an algebraically closed field is given by the following data:

a) $F_\tau =$ the set of branches of C passing through special points.
b) $V_\tau =$ the set of irreducible components of C, $g_v =$ the genus of the normalization of the component corresponding to v (which is sometimes denoted by C_v).
c) $\partial_\tau(f) = v$, iff the branch f of C belongs to the component C_v.

d) $j_\tau(f) = \overline{f}, f \neq \overline{f}$, iff the two branches $f, \overline{f}$ intersect at a common double point. In this way, the edges of the graph correspond bijectively to the special points of C.

e) $j_\tau(f) = f$, iff f is a branch passing through a labeled point of C. This yields a bijective correspondence between the labeled points of C and S which is isomorphic to the set of tails.

The combinatorial type of a curve will be the isomorphism class of its modular graph (τ, g).

1.2. Mumford classes. As previously mentioned, D. Mumford initiated the algebro–geometric study of the Chow ring of $\overline{M}_{g,0}$ in [Mu], where intersection theory of $\overline{M}_{g,n}$ is understood in the sense of orbifolds or stacks. To this end, he introduced certain classes, now called Mumford classes, whose definition was subsequently extended in [AC] to the spaces $\overline{M}_{g,n}$ with a slight alteration. We will use this version of the Mumford classes.

1.2.1. Definition. Let $p_n : \overline{C}_n \to \overline{M}_{g,n}$ be the universal curve, $x_i \subset \overline{C}_n, i = 1, \dots, n$ the images of the structure sections and $\omega_{C/M}$ the relative dualizing sheaf. Put for $a \geq 0$

$$\omega_n(a) = \omega_{g,n}(a) := p_{n*}(c_1(\omega_{C/M}(\sum_{i=1}^{n} x_i))^{a+1}) \in H^{2a}(\overline{M}_{g,n}, \mathbb{Q})^{\mathbb{S}_n} \tag{1.1}$$

where we used the notation of [KMK]; in [AC] these classes are denoted by κ_i. We will mostly omit g in our notation but not n.

The class $\omega_{g,n}(1)$ is actually $\frac{1}{2\pi^2}[v_{g,n}^{WP}]$ where $v_{g,n}^{WP}$ is the Weil–Petersson (see [Wo] and [Z]) 2–form so that

$$\int_{\overline{M}_{g,n}} \omega_{g,n}(1)^{3g-3+n} = (2\pi^2)^{3g-3+n} \times \text{WP–volume of } \overline{M}_{g,n} \tag{1.2}$$

(see [AC]).

1.2.2. Classical Weil–Petersson–volumes. The case of the classical WP–volumes (1.2) of the genus zero moduli spaces was first treated by P. Zograf [Z]. Put $v_3 = 1$ and

$$v_n := \int_{\overline{M}_{0n}} \omega_n(1)^{n-3}, \qquad n \geq 4\,. \tag{1.3}$$

The main result of [Z] then reads:

$$v_n = \frac{1}{2} \sum_{i=1}^{n-3} \frac{i(n-i-2)}{n-1} \binom{n-4}{i-1} \binom{n}{i+1} v_{i+2}\, v_{n-i}\,, \qquad n \geq 4\,. \tag{1.4}$$

If one considers the generating functions

$$\Phi(x) = \sum_{n=3}^{\infty} \frac{v_n}{n!(n-3)!}\, x^n\,, \quad h(x) = \Phi'(x) = \sum_{n=3}^{\infty} \frac{v_n}{(n-1)!(n-3)!}\, x^{n-1}\,, \tag{1.5}$$

then the recursion relation (1.4) directly translates into the differential equation

$$xh'' - h' = (xh' - h)\, h''\,. \tag{1.6}$$

for $h(x) = \Phi'(x)$.

The generating series (1.5) also arises in Liouville gravity models [Ma].

Differentiating the differential equation (1.6) once more, we obtain $h'h''' = xh''^3$. Setting $y = h'$ and interchanging the roles of x and y in the resulting cubic equation $yy'' = xy'^3$, it transforms into the Bessel equation

$$y\, \frac{d^2x}{dy^2} + x = 0.$$

This observation can be used to derive an explicit solution of (1.4) via an inverted modified Bessel function:

$$y = \sum_{n=3}^{\infty} \frac{v_n}{(n-2)!\,(n-3)!}\, x^{n-2} \quad \Longleftrightarrow \quad x = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!\,(m-1)!}\, y^m\,. \tag{1.7}$$

2. The moduli spaces $\overline{M}_{0n}$ of genus 0

Now, we turn to the genus 0 case which we will mainly focus on. We will briefly review the known results about the structure of the moduli spaces $\overline{M}_{0n}$, again quoting [M2], and then present our results. Most of the original work contained in the following section can be found in [Ke] and [Kn].

2.1. Theorem.

a) *For any $n \geq 3$, there exists a universal n–pointed stable curve $(\overline{\pi} : \overline{C}_{0n} \to \overline{M}_{0n}; x_i, i = 1, \ldots, n)$ of genus zero. This means that any such curve over a scheme T is induced by a unique morphism $T \to \overline{M}_{0n}$.*

b) *$\overline{M}_{0n}$ is a smooth irreducible projective algebraic variety of dimension $n-3$.*

c) *For any stable n–tree τ, there exists a locally closed irreducible subscheme $D(\tau) \subset \overline{M}_{0n}$, parameterizing exactly curves of the combinatorial type τ. Its codimension is equal to the cardinality $|E_\tau|$ of the set of edges. This subscheme depends only on the n isomorphism class of τ.*

d) *$\overline{M}_{0n}$ is the disjoint union of all $D(\tau)$. The closure of any of the strata $D(\tau)$ is the union of all the strata such that $\tau > \sigma$ in the sense of 2.2.4.*

2.2. Forgetful morphisms. Consider a stable pointed curve given by the data $(C, x_1, \ldots, x_{n+1})$. We say that $(C, x_1, \ldots, x_n)$ is obtained from the first curve by forgetting the point x_{n+1}. However, it might happen that the new curve is not stable any more. This is precisely the case, if the component of C supporting x_{n+1} has only one additional labeled point say x_j. In this situation, we can contract the unstable component to its intersection point x'_j with some other component. We call the resulting n–pointed curve $(C, x_1, \ldots, x_{j-1}, x'_j, x_{j+1}, \ldots, x_n)$ the result of stably forgetting x_{n+1}. Of course, regarding any S–curve, one can in the same manner stably forget the point x_s for any $s \in S$.

2.2.1. Theorem.

a) *There is a canonical flat and proper morphism $\pi_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0n}$ which acts on the isomorphism classes of $(n+1)$–pointed curves by stably forgetting the last point. More generally, there exists a canonical morphism $\pi_s : \overline{M}_{0S} \to \overline{M}_{0,S\setminus\{s\}}$ stably forgetting the point x_s of an S–pointed curve.*

b) *There exists a canonical isomorphism $\mu_n : \overline{M}_{0n+1} \to \overline{C}_{0,n}$.*

2.2.2. The dual tree of a genus 0 curve. Since we will be dealing with genus zero curves, the graphs which will be considered will be trees, i.e. connected and simply connected graphs.

2.2.3. Degeneration of genus 0 curves. Geometrically, a stable genus 0 curve can degenerate in the following way: given a partition of the set of special points of a component C_v it may split into two irreducible genus 0 curves where the special points are distributed in the way prescribed by the partition. This is the only codimension one specialization possible for pointed genus 0 curves.

On the level of trees, this corresponds to the addition of an edge at a given vertex v splitting the flags of this vertex into two sets corresponding to a partition of S.

2.2.4. Notation. Given a tree τ, a vertex $v \in V_\tau$ and a partition (F_1, F_2) of the set $F_\tau(v)$, we denote by $\tau_v^{F_1,F_2}$ the tree obtained by replacing the vertex v by an additional edge e which separates the flags of v according to the partition (F_1, F_2).

If a tree σ can be obtained from a tree τ by adding edges in the above manner, we will write $\tau > \sigma$.

2.3. Stratification. For any set S with $|S| \geq 3$, there exist the coarse moduli spaces $M_{0,S}$ and $\overline{M}_{0S}$ classifying irreducible respectively arbitrary stable S–marked curves. Furthermore, given any tree τ, one can consider the space

$$M_\tau := \Big(\prod_{v \in V_\tau} M_{0F_\tau(v)} \Big). \tag{2.1}$$

These spaces give a stratification of the compactified moduli space $\overline{M}_{0n}$ indexed by n–trees

$$\overline{M}_{0n} = \coprod_\tau \Big(\prod_{v \in V_\tau} M_{0F_\tau(v)} \Big). \tag{2.2}$$

3. Trees and the cohomology of the spaces $\overline{M}_{0S}$

One of the main tools for the study of the geometry of moduli spaces of curves is provided by the combinatorics associated to trees (respectively modular graphs). It is for instance possible, as already mentioned above, to give a stratification of $\overline{M}_{0S}$ in terms of S–trees. It is, in particular, also possible to present the cohomology of these spaces in terms of trees.

3.1. Spaces of Trees. Let k be a supercommutative $\mathbb{Q}$–algebra. Noticing that the stability condition implies that the number of isomorphism classes of stable S–trees is finite and that the maximal number of edges of a stable S–tree is $|S|-3$, we can define the graded free k–module over the set of isomorphism classes of stable S–trees via

$$\Gamma_{S,e} := \text{The set isomorphism classes of stable } S\text{–trees with } |E_\tau| = e \tag{3.1}$$

$$V(\Gamma_{S,e}) := \text{The free } k\text{–module over } \Gamma_{S,e} \tag{3.2}$$

$$V(\Gamma_S) := \bigoplus_{e=0}^{|S|-3} V(\Gamma_{S,e}). \tag{3.3}$$

3.2. Keel's presentation. As was shown in [Ke], the cohomology ring of $\overline{M}_{0S}$ can be presented in terms of classes of boundary divisors as generators and quadratic relations as introduced by [Ke]. The additive structure of this ring and the respective relations can then be naturally described in terms of stable trees (see [KM] and [KMK]).

More precisely: The boundary divisors of $\overline{M}_{0S}$ are in one–to–one correspondence with unordered 2–partitions $\{S_1, S_2\}$ of S, satisfying $|S_1| \geq 2$ and $|S_2| \geq 2$ (stability). Let $\{D_\sigma | \sigma = \{S_1, S_2\}$ a stable S–partition$\}$ be a set of commuting independent variables. Consider the ideal $I_S \subset F_S$ in the graded polynomial ring $F_S := k[D_{\{S_1,S_2\}}]$ generated by the following relations:

(i) $D_{\{S_1,S_2\}} D_{\{S_1',S_2'\}}$, if the number of non–empty pairwise intersections of these sets equals to 4.

(ii) $\forall$ distinct $i,j,k,l \in S : \sum_{ij\sigma kl} D_\sigma - \sum_{kj\tau il} D_\tau$

where the notation of the type $ij\sigma kl$ is used to imply that $\{i,j\}$ and $\{k,l\}$ are subsets of different parts of σ.

Set $H_S^* := F_S / I_S$.

3.2.1. Theorem [Ke]. *The map*

$$D_\sigma \longmapsto \begin{array}{l}\textit{dual cohomology class of the boundary divisor}\\ \textit{in } \overline{M}_{0S} \textit{ corresponding to the partition } \sigma\end{array} \tag{3.4}$$

induces the isomorphism of rings (doubling the degrees)

$$H_S^* \xrightarrow{\sim} H^*(\overline{M}_{0S}, k) \simeq A^*(\overline{M}_{0S})_k \tag{3.5}$$

where A^* is the Chow ring.

3.3. Additive structure of H_n^*. The additive structure of the cohomology can be nicely presented in terms of trees (see [KMK]). There (Proposition 1.3), it is proved that the set of trees with r edges is in bijection with the set of good monomials of degree r. We will briefly quote some of the notions and results from that paper. A monomial $D_{\sigma_1} \dots D_{\sigma_a} \in F_S$ is called good, if the family of 2–partitions $\{\sigma_1, \dots, \sigma_a\}$ is good, i.e. $a(\sigma_i, \sigma_j) = 3$, where for two unordered stable partitions $\sigma = \{S_1, S_2\}$ and $\tau = \{T_1, T_2\}$ of S

$$\begin{aligned} a(\sigma, \tau) := \quad & \text{the number of non–empty pairwise distinct sets} \\ & \text{among} \quad S_i \cap T_j,\ i, j = 1, 2. \end{aligned} \tag{3.6}$$

3.3.1. Lemma (1.2 of [KMK]). *Let τ be a stable S–tree with $|E_\tau| \geq 1$. For each $e \in E_\tau$ denote by $\sigma(e)$ the 2–partition of S corresponding to the one edge S–tree obtained by contracting all edges except for e. Then*

$$mon(\tau) := \prod_{e \in E_\tau} D_{\sigma(e)} \tag{3.7}$$

is a good monomial.

3.3.2. Proposition (1.3 of [KMK]). *For any $1 \leq r \leq |S| - 3$, the map $\tau \longmapsto mon(\tau)$ establishes a bijection between the set of good monomials of degree r in F_S and stable S–trees τ with $|E_\tau| = r$ modulo S–isomorphism. There are no good monomials of degree greater than $|S| - 3$.*

3.3.3. Additive relations. In [KMK] it is shown that the good monomials span the cohomology space and, furthermore, that all linear relations between them are generated by the relative versions of (ii);

$$\sum_{ij\tau'kl} mon(\tau') = \sum_{ik\tau''jl} mon(\tau'') \tag{3.8}$$

where $\{ij\tau'kl\}$ and $\{ik\tau''jl\}$ are the preimages of the contraction onto a given τ contracting exactly one edge onto a fixed vertex v separating the flags marked by i, j and k, l respectively i, k and j, l in such a way that they lie on different components after severing e where the markings i, j, k, l refer to flags which are part of the edges of the unique paths from v to the tails i, j, k, l in τ and it is required that the paths start along different edges.

3.3.4. Relations among trees. For any S–tree τ, any vertex v of τ and four distinct flags $f_i, f_j, f_k, f_l \in F_\tau(v)$ set

$$R_{\tau,v,(f_i,f_j,f_k,f_l)} = \sum_{\substack{(F_1,F_2) \\ f_i,f_j \in F_1 f_k,f_l \in F_2}} \tau_v^{F_1,F_2} - \sum_{\substack{(F_1,F_2) \\ f_i,f_k \in F_1 f_j,f_l \in F_2}} \tau_v^{F_1,F_2}. \tag{3.9}$$

Let

$$\mathcal{R}_S := span\{R_{\tau,v,(f_i,f_j,f_k,f_l)}\}. \tag{3.10}$$

Then we can reformulate Keel's Theorem together with the analysis of the additive structure of $H^*(\overline{M}_{0S})$ as follows

$$V(\Gamma_S)/\mathcal{R}_S \simeq A^*(\overline{M}_{0S}). \tag{3.11}$$

3.4. Forgetful morphisms and trees. The flat and proper morphisms $\pi_s : \overline{M}_{0S} \to \overline{M}_{0,S\setminus\{s\}}$ which forget the point marked by s and stabilize if necessary induce the maps π_* and π^* on the Chow rings where we omitted the subscript s which we will always do, if there is no risk of confusion.

We will now define the maps π_*, π^* on $V(\Gamma_S)$ corresponding under *mon* to the respective maps in the Chow rings of $\overline{M}_{0S}$ induced by the isomorphism (3.11).

Define π_* via

$$\pi_{s*}(\tau) = \begin{cases} \text{forget the tail number } s \text{ and stabilize, if the stabilization is necessary} \\ 0 \text{ otherwise} \end{cases} \tag{3.12}$$

For any S–tree τ and any $s \notin S$ set

$$\tau_v^s = \text{ the } (S \cup \{s\})\text{–tree obtained from } \tau \text{ by adding an additional tail marked by } s \text{ at the vertex } v. \tag{3.13}$$

Now define

$$\pi^{s*}(\tau) = \sum_{v \in V_\tau} \tau_v^s, \tag{3.14}$$

Taking the definition of *mon*, it is a straightforward calculation to check that indeed $mon(\pi^*(\tau)) = \pi^*(mon(\tau))$ and $mon(\pi_*(\tau)) = \pi_*(mon(\tau))$. Furthermore, one can check that these maps descend to the quotients by $\mathcal{R}_S$.

4. The intersection formula for strata classes

Starting from the tree description of the additive structure of the cohomology ring $H^*(\overline{M}_{0n})$, we give, as a first result, a formula for the intersection index of two strata classes. To this end, we first introduce the new notion of a good orientation and then provide the formula.

4.1. The intersection product for $H^*(\overline{M}_{0S})$. Consider the functional $\int_{\overline{M}_{0,S}} : H^*(\overline{M}_{0,S}) \to k$ is given by

$$mon(\tau) \longmapsto \begin{cases} 1, & \text{if} \quad \deg\, mon(\tau) = |S| - 3, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Notice that $\deg\, mon(\tau) = |S| - 3$ iff $|v| = 3$ for all $v \in V_\tau$, and $\overline{M}_\tau$ is a point in this case. We put $\langle \sigma_1, \sigma_2 \rangle = \int_{\overline{M}_{0S}} mon(\sigma_1) mon(\sigma_2)$ and set to calculate this intersection index for the case when $\deg\, mon(\sigma_1) + \deg mon(\sigma_2) = |S| - 3$. Generally, we will write $\langle m \rangle$ instead of $\int_{\overline{M}_{0S}} m$. We can assume that all pairs of different divisors of $mon(\sigma_1)$ and $mon(\sigma_2)$ are compatible, otherwise $\langle \sigma_1, \sigma_2 \rangle = 0$. Put $\tau = \sigma_1 \times \sigma_2$ in the category of S–morphisms. This is a tree with a marked subset of edges E corresponding to D_σ's whose squares divide $mon(\sigma_1) mon(\sigma_2)$. We denote by δ the subgraph of τ consisting of E and its vertices.

Consider an orientation of all edges of δ. Call it *good*, if for all vertices v of τ, the number of ingoing edges equals $|v| - 3$ where $|v|$ means the valence in τ. Notice that in case $v \notin V_\delta$, we interpret this as $|v| = 3$.

4.2. Proposition. *There cannot exist more than one good orientation of δ. If there is none, we have $\langle \sigma_1, \sigma_2 \rangle = 0$. If there is one, we have*

$$\langle \sigma_1, \sigma_2 \rangle = \prod_{v \in V_\tau} (-1)^{|v|-3} (|v| - 3)! \tag{4.2}$$

The proof can be found in the Appendix of [KMK]. It also follows from the more general Theorem 6.5 below.

4.3. Remark. The above proposition gives the intersection formula in terms of the strata–tree description. However, if one is interested in the intersection index of any number of strata classes or of some particular type of non–strata classes, one has to generalize Proposition 4.2. This will be the objective for the next sections. We will, in particular, provide the above mentioned intersection formula and, furthermore, present a basis of $H^*(\overline{M}_{0n})$ together with its intersection matrix and the inverse intersection matrix.

5. Partitions and Trees

In order to state the above mentioned theorem, we will first introduce some new notions to extend the combinatorics which can be handled by the trees introduced so far.

5.1. Trees with multiplicity. To generalize Proposition 4.2, we will have to deal with monomials which are not necessarily good; consequently, we will extend the notion of trees to that of trees with multiplicity which can be seen as the objects corresponding to arbitrary monomials in boundary divisors.

5.1.1. Definition. A S–tree with multiplicity is a pair (τ, m) consisting of a S–tree and a function $m : E_\tau \to \mathbb{N}$.

If no multiplicity function is given, we will assume that it is identically 1.

Call a monomial $D_{\sigma_1}^{m_1} \cdots D_{\sigma_k}^{m_k}$ nice, if $a(\sigma_i, \sigma_j) = 2$ or 3.

Set

$$mon((\tau, m)) := \prod_{e \in E_\tau} D_{\sigma(e)}^{m(e)}. \tag{5.1}$$

5.1.2. Proposition. *For any* $1 \le r \le |S|-3$*, the map:* $(\tau, m) \mapsto mon((\tau, m))$ *establishes a bijection between the set of nice monomials of degree* r *in* F_S *and stable* S*–trees with multiplicity* (τ, m) *with* $deg(\tau, m) := \sum_{e \in E_\tau} m(e) = r$.

Proof. Immediate from 3.3.2.

5.1.3. Remark. Notice that unlike in the case of good monomials it can happen that a nice monomial can represent a zero class, even if the degree is less or equal to $|S| - 3$.

5.2. Rooted trees and ordered partitions.

5.2.1. Remark. If we choose a distinguished element $s \in S$, we can define natural bijections between the following three sets:

a) unordered 2–partitions $\sigma = \{S_1, S_2\}$ of S
b) ordered 2–partitions $\sigma = \langle S_1, S_2 \rangle$ with the condition $s \in S_2$
c) subsets $T \subseteq S \setminus \{s\}$.

This is due to the fact that given the first component of an ordered pair of the above type the second one is uniquely determined.

5.2.2. The case of $\overline{n}$. In particular for $S = \overline{n}$, we choose n as the distinguished element and we equivalently index the generators of H^* by subsets $S \subset \overline{n-1}$ with the restriction $2 \leq |S| \leq n-2$ (note that this excludes the set $\overline{n-1}$ itself). We will denote the generator corresponding to such a set S:

$$D_S := D_{S,\overline{n}\setminus S},$$

for $S \subset \overline{n-1}$. The relations (i) and (ii) of Section 3.2 stated in this notation become:

(i') $D_S D_T$ if $S \cap T \neq \emptyset$ and the two sets do not satisfy an inclusion relation.
(ii') For any four numbers i, j, k, l:

$$\sum_{\substack{\overline{n-1}\supset T\supseteq\{i,j\}\\ k,l\notin T}} D_T + \sum_{\substack{\overline{n-1}\supset T\supseteq\{k,l\}\\ i,j\notin T}} D_T - \sum_{\substack{\overline{n-1}\supset T'\supseteq\{i,k\}\\ j,l\notin T}} D'_T - \sum_{\substack{\overline{n-1}\supset T'\supseteq\{j,l\}\\ i,k\notin T}} D'_T. \tag{5.2}$$

The expression for D_S^2 for a choice $i, j \in S$ and $k \notin S$ reads:

$$D_S^2 = -\sum_{S\supset T\supseteq\{i,j\}} D_S D_T - \sum_{\substack{S\subset T\subset\overline{n-1}\\ k\notin T}} D_S D_T. \tag{5.3}$$

This is the formula (5.4) from [KMK] with i, j, k, n playing the role of i, j, k, l.

The analogs of formula (3.8) follow in the same manner.

5.2.3. Rooted trees and orientation. A rooted S–tree will be a pair (τ, v_{root}) consisting of a S–tree τ and one of its vertices v_{root} called root. An orientation of a tree is considered to be a map $or : E_\tau \longrightarrow V_\tau$, with the restriction that e is incident to $or(e)$. We will use the terminology "e is pointing towards v" to indicate $v = or(e)$ ("pointing away" will be used on the same basis). The set $or^{-1}(v)$ will be called the incoming edges, the remaining incident edges will be considered as outgoing. Furthermore, notice that an oriented edge e of a tree defines a subtree by cutting e and selecting the tree containing $or(e)$. This subtree will be called the branch of e.

5.2.4. Natural orientation for a rooted tree. For a rooted tree (τ, v_{root}), there is a natural orientation defined by setting $or(e) =$ the vertex of e which is furthest away from the root (i.e. e is part of the unique path from this vertex to the root). Notice that in this orientation there is exactly one incoming edge to each vertex except for the root which has none. Therefore, the restriction of or induces an one–to–one correspondence of $V(\tau) \setminus \{v_{root}\}$ and $E(\tau)$.

$$\begin{aligned} e &\mapsto \text{vertex to which } e \text{ is pointing}\\ &\qquad \textit{inversely}\\ v &\mapsto \text{the unique incoming edge} \end{aligned} \tag{5.4}$$

5.2.5. Orientation for a n–tree. For a given n–tree, we will fix the root to be the vertex with the flag numbered by n emanating from it. This defines an one–to–one correspondence of n–trees with rooted n trees. Using this picture and Remark 5.2.1, we can equivalently view a n–tree (with multiplicity) as either given by the good (nice) collection of 2–partitions associated to its edges or as a good (nice) collection of subsets of $\overline{n-1}$ associated to its vertices. In the latter case, we associate to each vertex the set S of the 2–partition corresponding to the incoming edge, which *does not* contain n. In this way, denote for given nice σ and $S \in \sigma$ by v_S (respectively e_S) the vertex (respectively edge) corresponding to S.

Adopting this point of view, we can express quantities which are defined in the language of Remark 5.2.1 c) in terms of oriented n–trees. Let σ be a collection of stable subsets of $\overline{n}$, i. e. for each $S \in \sigma$ $S \subset \overline{n-1}$ and $|S| \geq 2$. Define for any $S \in \sigma$:

$$\begin{aligned} \omega_\sigma(S) &= \{T | T \subset S \text{ and maximal in this respect}\} \\ depth_\sigma(S) &= |\{T | T \in \sigma \text{ and } T \supseteq S\}| \end{aligned} \tag{5.5}$$

The definitions of (5.5) translate in the following way into tree language:

$$\begin{aligned} |S| &= |\{\text{tails marked by } i \in \overline{n-1} \text{ on the branch of } e_S\}| \\ \omega_\sigma(S) &= \{\text{outgoing edges of } v_S\} \\ depth_\sigma(S) &= \text{the distance from } v_S \text{ to } v_{root} \end{aligned} \tag{5.6}$$

where the distance is the number of edges along the unique shortest path.

6. The intersection form

6.1. Notation. Having introduced the notions of trees with multiplicity and good multiplicity orientations, we can now calculate the intersection form as a formula for two arbitrary monomials of boundary divisors of complementary degree. Recall that for a tuple (σ, m) of a nice collection of subsets of $\overline{n-1}$ and a multiplicity function $m : \sigma \mapsto \mathbb{N}$ we denote by $mon(\sigma, m)$ the monomial $\prod_{S\in\sigma} D_S^{m(S)}$. The *degree* of such a monomial is $\sum_{S\in\sigma} m(S)$. Furthermore, let $\tau(\sigma, m)$ be the tuple $(\tau(\sigma), m')$ where $\tau(\sigma)$ is the tree corresponding to the good monomial $\prod_{S\in\sigma} D_S$ and $m' : E_{\tau(\sigma)} \to \mathbb{N}$ is the the multiplicity function given by $e_S \to m(S)$.

6.2. Definition. A multiplicity orientation for a tree with multiplicity (τ, m) is a map $mult : F_\tau \setminus T_\tau \mapsto \mathbb{N}$ such that, if v_1 and v_2 are the vertices of an edge e:

$$mult((v_1, e)) + mult((v_2, e)) = m(e) - 1. \tag{6.1}$$

It is called good, if for every $v \in V_\tau$ it satisfies:

$$\sum_{f\in F_\tau(v)} mult(f) = |v| - 3. \tag{6.2}$$

This is the analog of the good orientation in [KMK].

6.3. Lemma. *For a n–tree (τ, m) in top degree (i.e. $\sum_{e\in E_\tau} m(e) = n - 3$), there exists, at most, one good multiplicity orientation.*

Proof. Assume that there are two good orientations $mult, mult'$. Consider the union of all edges on which $mult \neq mult'$. Each connected component of this union is a tree. Choose an end edge e of this tree and an end vertex v of e. At v, the sum over all flags f of $mult(f)$ and $mult'(f)$ must be equal, but on (v, e) these differ. Hence, there must exist an edge $e' \neq e$ incident to v upon which $mult((v, e'))$ and $mult'((v, e'))$ differ. But this contradicts to the choice of v and e.

The next lemma gives a way to decide whether this good multiplicity orientation exists and, if so, to calculate it.

6.4. Lemma. *Assume that an n–tree $\tau(\sigma, m)$ in top degree has a good multiplicity orientation mult. Let v_S be the vertex corresponding to $S \in \sigma$ and f_S be the flag of the unique incoming edge, then the following formula for its multiplicity holds:*

$$mult(f_S) = |S| - 2 - \sum_{T\in\sigma|T\subset S} m(T). \tag{6.3}$$

Proof. We will use induction on the distance from the end vertices (i.e those vertices with only one adjacent edge) in the natural orientation of n–trees given by 5.2.5; the case for the end vertices being trivial. Now let v_S be the vertex corresponding to S. By induction, we can assume that for all outgoing flags (6.3) holds; i.e. for all (v, e_T) with $T \in \omega_\sigma(S)$:

$$mult((v,e_T)) = m(T) - 1 - |T| + 2 + \sum_{T' \in \sigma | T' \subset T} m(T').$$

Inserting this into the condition (6.2), we arrive at

$$\begin{aligned} mult(f_S) &= |v_S| - 3 - \sum_{T \in \omega_\sigma(S)} mult((v,e_T)) \\ &= |S| - |\bigcup_{T \in \omega_\sigma(S)} T| + |\omega_\sigma(S)| - 2 - \sum_{T \in \omega_\sigma(S)} (m(T) - |T| + \sum_{T' \in \sigma | T' \subset T} m(T') + 1) \\ &= |S| - 2 - \sum_{T \in \sigma | T \subset S} m(T), \end{aligned} \tag{6.4}$$

where in the last step we have used $|\bigcup_{T\in\omega_\sigma(S)} T| = \sum_{T\in\omega_\sigma(S)} |T|$, since σ is a nice collection.

Consider again the functional $\int_{\overline{M}_{0,S}} : H^*(\overline{M}_{0,S}) \to k$ is given by

$$mon(\tau) \longmapsto \begin{cases} 1, & \text{if} \quad \deg mon(\tau) = |S| - 3, \\ 0 & \text{otherwise} \end{cases}$$

for any tree τ with $m \equiv 1$.

We put $\langle (\tau_1, m_1)\,(\tau_2, m_2) \rangle = \int_{\overline{M}_{0S}} mon((\tau_1,m_1)) mon((\tau_2,m_2))$ and set to calculate this intersection index for the case when the two classes are of complementary degree: $\deg(mon((\tau_1,m_1))) + \deg(mon((\tau_2,m_2))) = |S| - 3$. Generally, we will write $\langle \mu \rangle$ instead of $\int_{\overline{M}_{0S}} \mu$.

6.5. Theorem. *Let $mon(\sigma_1, m_1)$ and $mon(\sigma_2, m_2)$ be two monomials of complementary degree in H^*_n. If there is no good multiplicity orientation of $(\tau, m) := \tau(\sigma_1 \cup \sigma_2, m_1 + m_2)$, then $\langle mon(\sigma_1,m_1) mon(\sigma_2,m_2) \rangle = 0$. If there does exist one then:*

$$\langle mon(\sigma_1,m_1) mon(\sigma_2,m_2) \rangle = \prod_{v \in V_\tau} (-1)^{|v|-3} \frac{(|v|-3)!}{\prod_{f \in F(v)} (mult(f))!^2} \prod_{e \in E_\tau} (m(e)-1)!$$

where mult is the unique multiplicity orientation of (τ, m) provided by the Lemma 6.3 whose value is given in the formula (6.3).

Proof. Set $E := \{e \in E_\tau | m(e) > 1\}$ and δ the subtree consisting of E with multiplicity $m|_E$ and its vertices. Consider the canonical embedding $\varphi_\tau : \overline{M}_\tau \to \overline{M}_{0S}$.

$$\langle mon(\sigma_1,m_1) mon(\sigma_2,m_2) \rangle = \langle \prod_{e \in E} \varphi_\tau^*(D_{S(e)}^{m(e)-1}) \rangle \tag{6.5}$$

where the cup product in the r.h.s. is taken in $H^*(\overline{M}_\tau) \cong \otimes_{v \in V_\tau} H^*(\overline{M}_{0,F_\tau(v)})$. Applying an appropriate version of the formulas (5.2), we can write for any $e \in E$ with vertices v_1, v_2:

$$\varphi_\tau^*(D_{\sigma(e)}) = -\Sigma_{v_1,e} - \Sigma_{v_2,e}, \tag{6.6}$$

where

$$\Sigma_{v_i,e} \in H^*(\overline{M}_{0,F_\tau(v_i)}) \otimes \prod_{v\neq v_i}[\overline{M}_{0,F_\tau(v)}] \tag{6.7}$$

and $[\overline{M}_{0,F_\tau(v)}]$ is the fundamental class. Later, we will choose an expression for $\Sigma_{v_i,e}$ depending on the choice of flags denoted i,j or k,l in (5.2).

Inserting (6.6) into (6.5), we get

$$\langle mon(\sigma_1,m_1)mon(\sigma_2,m_2)\rangle = \sum_{or}\prod_{e\in E_\tau}(m(e)-1)!\langle \prod_{\substack{(v,e)\in F_\delta\\ or((v,e))\geq 1}} \frac{1}{or((v,e))!}(-\Sigma_{v,e})^{or((v,e))}\rangle \tag{6.8}$$

where *or* runs over all multiplicity orientations of δ. The summand of (6.8) corresponding to a given *or* can be non–zero only, if for every $v\in V_\delta$ the sum of the degrees of factors equals dim $\overline{M}_{0,F_\tau(v)} = |v|-3$. This is what was called a good multiplicity orientation. By Lemma 6.3 there can only exist one such orientation. Now assume that one good orientation *mult* exists. We can rewrite (6.8) as

$$\langle mon(\sigma_1,m_1)mon(\sigma_2,m_2)\rangle = \prod_{e\in E_\tau}(m(e)-1)! \prod_{\substack{(v,e)\in F_\delta\\ mult((v,e))\geq 1}} \frac{1}{mult((v,e))!}\langle(-\Sigma_{v,e})^{mult((v,e))}\rangle. \tag{6.9}$$

In view of (6.7), this expression splits into a product of terms computed in all $H^*(\overline{M}_{0,F_\tau(v)})$, $v\in V_\tau$ separately. Each such term depends only on $|v|$, and we want to demonstrate that it equals $(-1)^{|v|-3}\frac{(|v|-3)!}{\prod_{f\in F(v)}(mult(f))!}$. Put $|v|=m$, so $m\geq 3$. Let us identify F_τ with $\{1,\dots,m\}$ and denote by $D_\rho^{(m)}$ the class of a boundary divisor in $H^*(\overline{M}_{0,m})$ corresponding to a stable partition ρ of $\{1,\dots,m\}$ and set $d_i := mult((v,e_i))$ where e_i is the edge belonging to the flag $i\in\{1,\dots,m\}$. The contribution of v in (6.9) becomes

$$\prod_{i=1}^{m}\langle(-\Sigma_i^{(m)})^{d_i}\rangle := g(d_1,\dots,d_m) \tag{6.10}$$

where $-\Sigma_i^{(m)}$ is the element of (6.7) and the superscript (m) is again included to keep track of the spaces involved. We will prove the following properties of the function $g(d_1,\dots,d_m)$ identifying it as $(-1)^{m-3}\frac{(m-3)!}{d_1!\dots d_m!}$.

a) $g(0,0,0)=1$.
b) $g(d_1,\dots,d_m)$ is symmetric in the d_i.
c) If $d_m=0$, then
$$g(d_1,\dots,d_m) = -\sum_{i:d_i\geq 1} g(d_1,\dots,d_i-1,\dots,d_{m-1}).$$

6.5.1. Remarks. Notice that up to the minus sign in c) these are exactly the conditions satisfied by the numbers $\langle\tau_{\alpha_1}\ldots\tau_{\alpha_m}\rangle$ in genus zero [K2]. Furthermore, we can always choose the flags in such a way that the flags $1,\ldots,k$ ($k\le m-3$) belong to the edges e with $mult(f(v,e))\ge 1$.

ad a) We have by definition $\langle[\overline{M}_{0,3}]\rangle=1$.

ad b) The symmetry results from the fact that the integral in question does not depend on a renumbering of the flags.

ad c) First, we can use relation (5.2) for any k,l to write

$$-\Sigma_i^{(m)}=\sum_{\rho:\ i\rho\{k,l\}}-D_\rho^{(m)}. \tag{6.11}$$

We will calculate (6.10) inductively. Consider the projection map (forgetting the (m)–th point) $p:\ \overline{M}_{0,m}\to\overline{M}_{0,m-1}$ and the i–th section map $x_i:\ \overline{M}_{0,m-1}\to\overline{M}_{0,m}$ obtained via the identification of $\overline{M}_{0,m+3}$ with the universal curve. We have $p\circ x_i=\mathrm{id}$, and x_i identifies $\overline{M}_{0,m-1}$ with $D_{\sigma_i}^{(m)}$ where

$$\sigma_i=\{\{m,i\}\{1,\ldots,\widehat{i},\ldots,m-1\}\};$$

so if we choose some $k,l\neq m$:

$$\sum_{\rho:\ i\rho\{k,l\}}-D_\rho^{(m)}=-p^*\Big(\sum_{\rho':\ i\rho'\{k,l\}}D_{\rho'}^{(m-1)}\Big)-x_{i*}([\overline{M}_{0,m-1}]). \tag{6.12}$$

We will now replace one of the Σ_i for each i with $d_i\ge 1$ using (6.11) with some arbitrary $k,l\neq m$. Then (6.10) reads

$$\prod_{i=1}^m\langle\Big(-p^*(\sum_{\rho':\ i\rho'\{k,l\}}D_{\rho'}^{(m-1)})-x_{i*}([\overline{M}_{0,m-1}])\Big)(-\Sigma_i^{(m)})^{d_i-1}\rangle \tag{6.13}$$

where ρ' runs over stable partitions of $\{1,\ldots,m-1\}$. We represent the resulting expression as a sum of products consisting of several p^*–terms and several x_{i*}–terms each. If such a product contains two or more x_{i*}–terms, it vanishes, because the structure sections pairwise do not intersect. We obtain

$$\sum_{i:d_i\ge1}\langle\prod_{j\neq i:d_j\ge1}\Big(-p^*(\sum_{\rho':\ j\rho'\{k,l\}}D_{\rho'}^{(m-1)})(-\Sigma_j^{(m)})^{d_j-1}\Big)(-x_{i*}([\overline{M}_{0,m-1}]))(-\Sigma_i^{(m)})^{d_i-1}\rangle$$
$$+\langle\prod_{i:d_i\ge1}p^*(-\sum_{\rho':\ i\rho'\{k,l\}}D_{\rho'}^{(m-1)})(-\Sigma_j^{(m)})^{d_j-1}\rangle. \tag{6.14}$$

If $d_i-1>0$, then the summand containing an x_{i*}–term will vanish. To see this, again replace one of the Σ_i using (6.11), but with $k=m$ and some l. In case $d_i-1=0$, we can write the respective term in the sum in (6.14) as

$$\langle(p^*(-\sum_{\rho':\ j\rho'\{k,l\}}D_{\rho'}^{(m-1)})^{d_j-1})(-x_{i*}([\overline{M}_{0,m-1}]))\rangle$$

by replacing the Σ_j according to (6.12) and again using the fact that the structure sections do not pairwise intersect. Using induction on the last summand in (6.14), we arrive at the situation where all $\Sigma_i^{(m)}$'s have been replaced. And the product only contains $p^*(\Sigma_i^{(m-1)})$–terms, but this term vanishes, because dim $\overline{M}_{0,m-1} = m-2$. Finally, we are left with one summand for each $i : d_i \geq 1$ containing only one x_{i*}–term and p^*–terms. Using the projection formula

$$\langle p^*(X)x_{i*}([\overline{M}_{0,m-1}])\rangle = \langle X\rangle$$

one sees that each such term equals $-g(d_1, \ldots d_i - 1, \ldots, d_{m-1})$. And the result follows.

6.6. Remark. Another approach to the theorem above is given by excess intersection theory [F] using the formula for the normal bundle for strata given in [HL]; as pointed out by E. Getzler who used a modular graph version of our trees with multiplicity in calculations pertaining to the case of $g = 1$ [Ge2].

7. A boundary divisorial basis and its tree representation

The work presented in this section is inspired by the presentation of a basis of the cohomology ring of $\overline{M}_{0n}$ given in terms of hyperplane sections in [Yu] who worked out a basis in another presentation of the cohomology ring developed by DeConcini and Procesi [CP] via hyperplane arrangements. Especially the notion of the *–operation and the partial order have been adapted from [Yu] to the present context.

7.1. Preliminaries. In order to state the basis, we make use of certain classes

$$D_S x_S^k := \pi_{f_S *}(D_S^{k+1} D_{S \amalg f_S}), \quad k \geq 0 \tag{7.1}$$

where $\pi_{f_S *} : \overline{M}_{0,\overline{n} \amalg f_S} \to \overline{M}_{0n}$ is the forgetful map forgetting the point f_S.

Another way to present these classes is given by the following observation. Consider the following decomposition of D_S^2 using (5.3):

$$D_S^2 = D_S(-\sum_{\{i,j\} \subset T \subset S} D_T - \sum_{\substack{\overline{n-1} \supset T' \supset S \\ k \notin T'}} D'_T) =: D_S(x_S + y_S) \tag{7.2}$$

for any choice of $i, j \in S, k, l \notin S$. With the notation (7.2), we can write D_S^{k+1} in the same spirit as:

$$D_S^{k+1} = D_S(\sum_{i=0}^{k} \binom{k}{i} x_S^i y_S^{k-i}). \tag{7.3}$$

In the context of the proof of Theorem 6.5, each summand of (7.3) corresponds to a choice of multiplicity orientation. In particular, the term with x_S^i corresponds to the one which satisfies $mult(f_S) = i, mult(f_{S^c}) = k - i$ for the flags f_S and f_{S^c} of e_S, so that we can identify (7.1) with the summand corresponding to $mult(f_S) = k, mult(f_{S^c}) = 0$.

7.1.1. A tree representation. A tree representation for a class (7.1) is given by a choice of an ordered $k+1$ element subset $\langle f_1, \ldots, f_{k+1} \rangle$ of S as the sum over all assignments of the flags of $S \setminus \{f_1, \cdots, f_{k+1}\}$ to the vertices of the linear tree determined by the monomial $D_{\{f_1,f_2\}} D_{\{f_1,f_2,f_3\}} \ldots D_{\{f_1,\ldots,f_{k+1}\}}$

$$D_S x_S^k = (-1)^k D_S \sum_{\substack{\langle S_1,\ldots,S_k \rangle \\ S_1 \amalg \cdots \amalg S_k = S \setminus \{f_1,\ldots,f_{k+1}\}}} D_{\{f_1,f_2\} \amalg S_1} D_{\{f_1,f_2,f_3\} \amalg S_2} \ldots D_{\{f_1,\ldots,f_{k+1}\} \amalg S_k} \tag{7.4}$$

or more generally, let τ given by $D_{T_1} \cdots D_{T_k}$ be any tree with $|v_{T_i}| = 3$ for $i = 1, \ldots, k$ and $T_1 \cup \cdots \cup T_k = \{f_1, \ldots, f_{k+1}\}$ then

$$D_S x_S^k = (-1)^k D_S \sum_{\substack{\langle S_1,\ldots,S_k \rangle \\ S_1 \amalg \cdots \amalg S_k = S \setminus \{f_1,\ldots,f_{k+1}\}}} D_{T_1 \amalg S_1} \ldots D_{T_k \amalg S_k}. \tag{7.5}$$

Both (7.4) and (7.5) follow from (5.2) with the appropriate choices for the flags.

7.2. The basis. Consider a class of the following type

$$\mu = \pi_{n*}(D_{S_1} x_{S_1}^{m(S_1)} \cdots D_{S_k} x_{S_k}^{m(S_k)} D_{\overline{n-1}} x_{\overline{n-1}}^{m(\overline{n-1})}), \quad m(S) \geq 0. \tag{7.6}$$

To this class we associate the underlying $(n+1)$–tree $\tau(\mu)$ determined by the monomial $D_{S_1} \dots D_{S_k} D_{\overline{n-1}}$. The powers $m(S)$ then determine a unique multiplicity orientation in the sense of 7.1 given by $mult(f_S) := m(S), mult(f_{S^c}) = 0$ where f_S and f_{S^c} are the flags corresponding to the edge e_S in $\tau(\mu)$.

Using the equations of the type (7.4), we can associate to each monomial μ a sum of good monomials which we will call $tree(\mu)$.

Consider the following set:

$$\mathcal{B}_n := \{\pi_{n+1*}(D_{S_1} x_{S_1}^{m(S_1)} \cdots D_{S_k} x_{S_k}^{m(S_k)} D_{\overline{n-1}} x_{\overline{n-1}}^{m(\overline{n-1})}) \mid 0 \leq m(S) \leq |v_S| - 4 \text{ and } 0 \leq m(\overline{n-1}) \leq |v_{\overline{n-1}}| - 3\}. \tag{7.7}$$

7.2.1. Proposition. *The set $\mathcal{B}_n$ is a basis for $A^*(\overline{M}_{0n})$.*

Proof. By Lemma 7.2.2 and 7.2.7.

7.2.2. Lemma. *The set $\mathcal{B}_n$ spans $A^*(\overline{M}_{0n})$.*

Proof. From [Ke] and [KMK] we know that the good monomials span; so it will be sufficient to show that any such monomial is in the span of $\mathcal{B}_n$. Now let $\tau(\mu)$ be the tree corresponding to such a good monomial μ. If for all $v \in V_\tau$ $|v| \geq 4$, then the monomial is already in $\mathcal{B}_n$. If not let τ_3 be a maximal subtree of τ whose vertices, except for the root (induced by the natural orientation), all have valence three; call such a tree a 3–subtree and the number of its edges its length. Furthermore, let R be the set associated with the root. Let $F_3(\tau_3)$ be the set of tails of τ_3 without the ones coming from the root. The formula (7.5) for the tree representation of $D_R x_R^l$ with the choice of $F_3(\tau_3)$ as the fixed set and τ_3 as a 3–subtree expresses τ in terms of trees with less maximal 3–subtrees of maximal length whose vertices either comply with the conditions of $\mathcal{B}_n$ or are part of a unique maximal subtree whose root v_R has multiplicity 0, i.e. x_R does not divide the monomial corresponding to the tree. Notice that if the root v_R of any 3–subtree is three–valent then $R = \overline{n-1}$. We can now proceed by induction of the number of such maximal 3–subtrees with the maximal number of edges l.

7.2.3. The *–operation. We define the following involution on $\mathcal{B}_n \times \mathbb{Z}_2$:

$$\begin{aligned}\pi_{n+1*}(D_{S_1}x_{S_1}^{m(S_1)} \cdots D_{S_k}x_{S_k}^{m(S_k)} D_{\overline{n-1}}x_{\overline{n-1}}^{m(\overline{n-1})}) &\xrightarrow{*} \\ \pi_{n+1*}((-1)^{|v_{S_1}|-3}D_{S_1}x_{S_1}^{|v_{S_1}|-4-m(S_1)} \cdots &(-1)^{|v_{S_k}|-3}D_{S_k}x_{S_k}^{|v_{S_k}|-4-m(S_k)} \\ &(-1)^{|v_{\overline{n-1}}|-3}D_{\overline{n-1}}x_{\overline{n-1}}^{|v_{\overline{n-1}}|-3-m(\overline{n-1})}).\end{aligned} \tag{7.8}$$

This operation preserves the underlying tree $\tau(\mu)$, but changes the multiplicities in such a way that μ and μ^* have complementary dimensions. More precisely, consider μ as the push forward of the class $\bigotimes_{v_S \in V_{\tau(\mu)}} x_S^{m(S)} \in H^*(\overline{\mathcal{M}}_{\tau(\mu)})$ to $H^*(\overline{M}_{0n})$, then, locally at each vertex, we have a class of degree $m(S)$. This class is replaced under the *–operation by a "dual" class of complementary degree $\dim(\overline{M}_{0,F_\tau}(v_S)) - m(S)$ which is provided as a summand of $\varphi_{D_S}^*(D_S x_S^{|v_S|-4-m(S)})$.

7.2.4. Lemma. *For two elements μ, ν of $\mathcal{B}_n$ the integral $\int_{\overline{M}_{0n}} \mu\nu^*$ does not vanish iff $\tau(\mu\nu^*)$ is nonzero and if there is one good multiplicity orientation among the multiplicity orientations satisfying $mult(f_S) = m^\mu(S) + m^{\nu^*}(S) + 1, mult(f_{S^c}) = 0$ or $mult(f_S) = m^\mu(S) + m^{\nu^*}(S), mult(f_{S^c}) = 1$ where f_S, f_{S^c} are the flags of the edge e_S. If such an orientation exists, it is unique and*

$$\int_{\overline{M}_{0n}} \mu\nu^* = \prod_{v \in V_{\tau(\nu)}} (-1)^{|v|-3} \prod_{v \in V_{\tau(\mu\nu^*)}} (-1)^{|v|-3} \frac{(|v|-3)!}{\prod_{f \in F_{\tau(\mu\nu^*)}(v)} (mult(f))!}. \tag{7.9}$$

Proof. The formula (7.9) and the conditions for μ and ν as well as the ones for the considered multiplicity orientations follow from Theorem 6.5 by considering the summands of

$$\pi_{n+1*}(D_{S_1}^{\epsilon(S_1)+m(S_1)} \cdots D_{S_l}^{\epsilon(S_l)+m(S_l)} D_{\overline{n-1}}^{m(\overline{n-1})}),$$

corresponding via 7.1 to the given monomial

$$\mu\nu^* = \pi_{n+1*}(D_{S_1}^{\epsilon(S_1)}x_{S_1}^{m(S_1)} \cdots D_{S_l}^{\epsilon(S_l)}x_{S_l}^{m(S_l)} D_{\overline{n-1}}x_{\overline{n-1}}^{m(\overline{n-1})})$$

with $\epsilon(S) \in \{1,2\}$.

Notice that in the formula (7.9) the binomial coefficients $\binom{m(e_S)-1}{mult(f_S)}$ which appear in Theorem 6.5 are absent. This is due to the fact that these factors stemming from the expansion of $D_S^{m(e_S)}$ as in (7.3) are stripped off in the definition of the classes $D_S x_S^k$.

7.2.5. An order. Given two monomials μ, μ' of type (7.6) of the same degree we write $\mu \prec \mu'$, if for the maximal integer k such that all sets of the depth d vertices for $1 \le d \le k$ coincide and $m(S) = m'(S)$ for all sets of the depth d' vertices for $1 \le d' < k$ one of the following conditions holds

(a) $m(S) \le m'(S)$ for all S of depth k and the inequality is strict for at least one S or
(b) $m(S) = m'(S)$ and $|v_S| \ge |v'_S|$ for all S of depth k and there is at least one S where the inequality is strict.

It is easy to check that this defines a partial order on $\mathcal{B}_n$.

The *–operation connects with the partial order $\prec$ in the following way:

7.2.6. Lemma. *If $\mu, \nu \in \mathcal{B}_n$ are two distinct basis elements ($\mu \neq \nu$) and $\mu\nu^* \neq 0$, then $\mu \prec \nu$.*

Proof. We will use superscripts μ, ν to refer to the quantities concerning the monomials μ, ν and take quantities without any superscript to refer to $\mu\nu^*$. So the notation $|v_S^\nu|$ is used for the valence of the vertex v_S in the tree $\tau(\nu)$ and $|v_S|$ without any superscript is taken to be the valence of the vertex v_S in the tree $\tau(\mu\nu^*)$. If $\mu\nu^* \neq 0$ then the underlying tree of $\mu\nu^*$ carries a unique good multiplicity orientation by Theorem 6.5. Furthermore, the underlying trees of μ and ν coincide up to depth k; this is the first condition for k. From this, together with Lemma 7.2.3, it follows that the good multiplicity orientation up to depth $k-1$ is given by $mult(f_S) = |v_S| - 3$. Now at depth k we must have $mult(f_S) \leq |v_S| - 3$ and, because the multiplicity orientation is fixed for all lower depths as specified, we also have $mult(f_S) = m^\mu(S) + m^{\nu^*}(S) + \delta_{S,\overline{n-1}} = m(S) + |v_S^\nu| - 3 - m^\nu(S)$. Combining these two relations, we find the condition:

$$m^\mu(S) - m^\nu(S) \leq |v_S| - |v_S^\nu|. \tag{7.10}$$

Furthermore, we have the inequalities $|v_S| \leq |v_S^\nu|, |v_S| \leq |v_S^\mu|$, since $\tau(\mu)$ and $\tau(\nu^*) = \tau(\nu)$ result from $\tau(\mu\nu^*)$ via contractions of edges which only increase the number of flags at the remaining vertex. So the left–hand side of (7.10) is less or equal to zero:

$$m^\mu(S) - m^\nu(S) \leq 0. \tag{7.11}$$

Thus, if the inequality is strict for some S, we arrive at condition (a), if, however, $m^\mu(S) = m^\nu(S)$ for all S of depth k, the following inequality must also hold:

$$0 \leq |v_S^\mu| - |v_S^\nu|. \tag{7.12}$$

Equality for all S in (7.12), however, would contradict the choice of k, since if $m_S^\mu = m_S^\nu$ and $|v_S^\mu| = |v_S^\nu|$, we have $|v_S^\nu| = |v_S| = |v_S^\mu|$ from the above inequalities so that there are no contractions from $\tau(\mu\nu^*)$ to $\tau(\mu)$ and $\tau(\nu)$ up to depth $k+1$ and the sets of depth $k+1$ corresponding to the outgoing edges of v_S^μ and v_S^ν must also coincide.

7.2.7. Lemma. *Consider the matrix $T = (t_{\mu,\nu})_{\mu,\nu\in\mathcal{B}_n}$ given by*

$$t_{\mu,\nu} := \int_{\overline{M}_{0n}} \mu\nu^*. \tag{7.13}$$

This matrix is unipotent and the entry $t_{\mu,\nu}$ is determined by Lemma 7.2.3. In particular, the set $\mathcal{B}_n$ is linear independent.

Proof. For the diagonal entries $\int \mu\mu^*$ $mult(f_S) = |v_S| - 3$ is a good multiplicity orientation so that (7.9) renders $t_{\mu\mu^*} = 1$. Furthermore, by considering any extension of the partial order to a total order, the unipotency is proved by Lemma 7.2.6.

7.3. The intersection form and its inverse for the basis $\mathcal{B}_n$. With the help of the matrix T introduced in 7.2.6, we can write the matrix M for the intersection form in the basis $\mathcal{B}_n$ as $M = TP$ where the matrix P is the matrix representation of the *–operation given by the signed permutation matrix

$$P_{\mu,\nu} = (-1)^{n-2-|E_{\tau(\mu)}|}\delta_{\mu,\mu^*}. \tag{7.14}$$

7.3.1. Theorem. *The Gram–matrix $(m_{\mu\nu})$ for the basis $\mathcal{B}_n$ is given by*

$$m_{\mu\nu} = (-1)^{n-2-|E_{\tau(\nu)}|}t_{\mu\nu^*} \tag{7.15}$$

and its inverse matrix $(m^{\mu\nu})$ is given by the formula:

$$m^{\mu\nu} = (-1)^{n-2-|E_{\tau(\mu)}|}(\delta_{\mu^*\nu} + \sum_{k\geq 0}(-1)^{k+1}\sum_{\mu^*\prec\tau_1\cdots\prec\tau_k\prec\nu} t_{\mu^*\tau_1}t_{\tau_1\tau_2}\cdots t_{\tau_{k-1}\tau_k}t_{\tau_k\nu}) \tag{7.16}$$

where the values for the $t_{\sigma,\sigma'}$ are given by (7.9) and the sum over k is finite.

Proof. The formula (7.15) follows from the above decomposition $M = TP$. To prove the formula (7.16), set $N := id - T$. According to Lemma 7.2.6, N is nilpotent and the inverse to the intersection form can now be written as

$$M^{-1} = PT^{-1} = P(id + N + N^2 + \dots) \tag{7.17}$$

where the sum in (7.17) is finite.

7.3.2. Corollary. In the notation of Theorem 7.3.1, the diagonal class $\Delta_{\overline{M}_{0n}}$ of $\overline{M}_{0n}\times\overline{M}_{0n}$ has the following representation in $H^{2(n-3)}(\overline{M}_{0n}\times\overline{M}_{0n})$:

$$\Delta_{\overline{M}_{0n}} = \sum_{\mu,\nu\in\mathcal{B}_n}\mu\, m^{\mu\nu}\otimes\nu. \tag{7.18}$$

7.4. Particular cases. Writing down the results of this and the previous section, we obtain the following intersection matrices M_n for small values of n:

$n=3$ $M_3 = (1)$.

$n=4$ For the basis $\pi_{5*}(D_{1,2,3}), \pi_{5*}(-D_{1,2,3}x_{1,2,3})$ we obtain

$$M_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$n=5$ For the basis $\pi_{6*}(D_{1,2,3,4}), D_{1,2,3}, D_{1,2,4}, D_{1,3,4}, D_{2,3,4}, \pi_{6*}(D_{1,2,3,4}x_{1,2,3,4})$, $\pi_{6*}(D_{1,2,3,4}x^2_{1,2,3,4})$ the intersection matrix is:

$$M_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$n = 6$ In this case, the intersection matrix also has only nonzero entries for the integrals of dual classes under the $*$–operation: $\int_{\overline{M}_{0n}} \mu\mu^*$ whose values are $(-1)^{4-|E_{\tau(\mu)}|}$.

$n \geq 7$ For the higher values of n, the structure of the matrix T is not diagonal, since entries other than those coming from the product of *–dual classes can also be nonzero, e.g. $\langle D_{i,j,k,l}x_{i,j,k,l}D_{i,j,k,l}x_{i,j,k,l}\rangle$ in $\overline{M}_{0,7}$. Thus, the *–operation fails to give the Poincaré duality for these spaces.

However, on the subspace $A^1(\overline{M}_{0n}) \oplus A^{n-4}(\overline{M}_{0n})$, the *–operation does provide the Poincaré duality as can be deduced from Lemma 7.2.3. On this subspace, the matrix T is just the identity matrix so that the restriction to this subspace of M_n is given by P. In the case of small $n < 7$ this subspace is already the whole space so that the matrices in the previous cases are just given by P.

8. Higher Weil–Petersson–volumes

As a second part in the study of the geometry of the moduli spaces of curves, we will now consider —as an extension of (1.2)— the integrals of the type

$$\int_{\overline{M}_{g,n}} \omega_{g,n}(1)^{m(1)} \dots \omega_{g,n}(a)^{m(a)} \dots , \qquad \sum_{a \geq 1} am(a) = 3g - 3 + n \tag{8.1}$$

which we will call *higher WP–volumes*. In [KMZ] several formulas for these volumes and their generating functions were derived. For genus zero, we proved a recursive formula whose proof, which uses calculations of the $\omega_n(a)$ via strata classes, will be reproduced below.

Among the further results of the collaboration [KMZ] is a closed formula which expresses each higher volume in genus zero as an alternating sum of multinomial coefficients and a higher genus generalization of this formula in which the multinomial coefficients are replaced by the correlation numbers $\langle \tau_{d_1} \dots \tau_{d_n} \rangle$ which are computable via Witten–Kontsevich's theorem [W], [K1].

Encoding the values of the higher WP–volumes into a generating function in infinitely many variables, one can translate the recursion relation into an infinite system of non–linear differential equations for this generating function. Using a slightly modified version of this generating function, it is shown in [KMZ] that the above system of non–linear differential equations turn into a linear system for its inverse power series which can be solved explicitly. This fact will be used in the next Chapter.

Recently, the $g = 1$ case has been treated in [KK] following the lines of [KMZ].

We will now reproduce the proof of the recursive formula and quote the other results from [KMZ].

8.1. Recursive relations for the generating function.

8.1.1. Notation. Let N^∞ be the semigroup of sequences of the form $\mathbf{m} = (m(1), m(2), \dots)$ where the $m(a)$ are nonnegative integers and $m(a) = 0$ for sufficiently large a.

Set

$$V_g(\mathbf{m}) := \frac{1}{(\sum_{a \geq 1} am(a))!} \int_{\overline{M}_{g,n}} \prod_{a \geq 1} \frac{\omega_{g,n}(a)^{m(a)}}{m(a)!} \in \mathbb{Q} \tag{8.2}$$

where the r.h.s. is interpreted as zero unless $\sum_{a \geq 1} am(a) = \dim \overline{M}_{g,n} = 3g-3+n$. In the rest of this section $g = 0$ and $V_0(\mathbf{m})$ is simply denoted by $V(\mathbf{m})$.

To shorten expressions like (8.2), a shorthand notation of the following type is utilized

$$\begin{gathered} |\mathbf{m}| := \sum_{a \geq 1} am(a), \quad \|\mathbf{m}\| := \sum_{a \geq 1} m(a), \quad \mathbf{m}! := \prod_{a \geq 1} m(a)!, \\ \boldsymbol{\omega}_n^{\mathbf{m}} = \prod_{a \geq 1} \omega_{g,n}(a)^{m(a)}, \quad \mathbf{s}^{\mathbf{m}} = \prod_{a \geq 1} s_a^{m(a)} \end{gathered} \tag{8.3}$$

where $\mathbf{s} = (\mathbf{s_1}, \mathbf{s_2}, \dots)$ is a family of independent formal variables or complex numbers. For instance, $V(\mathbf{m}) = \int \boldsymbol{\omega}^{\mathbf{m}}/\mathbf{m}!\,|\mathbf{m}|!$ in this notation.

8.1.2. A recursive formula for $V(\mathbf{m})$. Put

$$K(n_1, \dots, n_a) := \frac{1}{n_1(n_1+n_2)\cdots(n_1+\cdots+n_a)} \tag{8.4}$$

and denote by $\delta_a \in N^\infty$ the sequence with 1 at the a–th place and zeros elsewhere.

8.1.3. Theorem. *For any* $\mathbf{m}$ *and* $a \geq 1$, *we have:*

$$(m(a)+1)V(\mathbf{m}+\delta_a) = (|\mathbf{m}|+a+1) \sum_{\mathbf{m}=\sum_{i=1}^{a+1} \mathbf{m}_i} K(n_1, \dots, n_a) \prod_{i=1}^{a+1} V(\mathbf{m}_i) \tag{8.5}$$

where in each summand of (8.5)

$$(n_1, \dots, n_a) := (|\mathbf{m}_1|, \dots |\mathbf{m}_a|) + (2, 1, \dots, 1) \tag{8.6}$$

*(*notice the absence of $|\mathbf{m}_{a+1}|$). *These relations uniquely define* $V(\mathbf{m})$, *starting with* $V(0) = 1$.

8.1.4. A particular case of (8.5). Applying (8.5) to $V(m) := V(m, 0, 0, \dots)$ and $a = 1$ yields:

$$\begin{aligned}(m+1)V(m+1) &= (m+2) \sum_{m=m_1+m_2} \frac{1}{m_1+2} V(m_1)V(m_2) \\ &= \frac{1}{2}(m+2) \sum_{m=m_1+m_2} (\frac{1}{m_1+2} + \frac{1}{m_2+2})\, V(m_1)\, V(m_2) \end{aligned} \tag{8.7}$$

so that

$$\frac{V(m+1)}{m+3} = \frac{1}{2} \frac{(m+2)(m+4)}{(m+1)(m+3)} \sum_{m=m_1+m_2} \frac{V(m_1)}{m_1+2} \frac{V(m_2)}{m_2+2}.$$

On the other hand, Zograf's recursive relations (1.4) can be rewritten as

$$\frac{(n-2)v_n}{(n-3)!(n-1)!} = \frac{1}{2} \frac{(n-2)n}{(n-3)(n-1)} \sum_{\substack{n+2=p+q \\ p,q \geq 3}} \frac{(p-2)v_p}{(p-3)!(p-1)!} \frac{(q-2)v_q}{(q-3)!(q-1)!}.$$

These relations agree for $V(n-3) = v_n/(n-3)!^2$ which is the correct formula in view of (1.3) and (8.2).

We will now start proving Theorem 8.1.3.

For any stable n–tree σ, we put (with notation (8.3))

$$\Omega_n(\mathbf{m}, \sigma) = \int_{\overline{M}_\sigma} \frac{\varphi_\sigma^*(\boldsymbol{\omega}_n^{\mathbf{m}})}{\mathbf{m}!}, \tag{8.8}$$

interpreting this as zero unless $n - 3 - |\mathbf{m}| = \operatorname{codim} \varphi_\sigma(\overline{M}_\sigma) = |E_\sigma|$. If σ_n is an one–vertex n–tree, we write $\Omega_n(\mathbf{m}) := \Omega_n(\mathbf{m}, \sigma_n)$. Notice that $\Omega_n(a)$ from

[KMK] is $\left(\frac{n-3}{a}\right)!\,\Omega_n(\frac{n-3}{a}\delta_a)$ in our present notation. The numbers $V(\mathbf{m})$ in (8.2) are $\Omega_n(\mathbf{m})/|\mathbf{m}|!$.

8.2. Lemma. *We have*

$$\Omega_n(\mathbf{m},\sigma) = \sum_{\substack{(\mathbf{m}_v|v\in V_\sigma):\\ \mathbf{m}=\sum \mathbf{m}_v}} \prod_{v\in V_\sigma} \Omega_{|v|}(\mathbf{m}_v), \tag{8.9}$$

where the sum in r.h.s. is taken over all partitions of $\mathbf{m}$ *indexed by vertices of* σ.

Proof. This follows from the fact that the $\omega_n(a)$ form a "logarithmic CohFT", a notion which was defined in [KMK]. This means that they satisfy a certain additivity property which was established in [AC], (8.8). For any genus:

$$\varphi_\sigma^*(\omega_n(a)) = \sum_{w\in V_\sigma} \mathrm{pr}_w^*(\omega_{|w|}(a)) \tag{8.10}$$

where $\overline{M}_\sigma$ is identified with $\prod_{w\in V_\sigma} \overline{M}_{0,|w|}$ and pr_w is the respective projection. Although these identifications are defined only up to the action of $\prod_{w\in V_\sigma} \mathbb{S}_{|w|}$, the classes $\mathrm{pr}_w^*(\omega_{|w|}(a))$ are well defined since they are $\mathbb{S}_{|w|}$–invariant.

Hence

$$\begin{aligned}
\int_{\overline{M}_\sigma} \varphi_\sigma^*(\boldsymbol{\omega}_n^{\mathbf{m}}) &= \int_{\prod_{v\in V_\sigma} \overline{M}_{0,|v|}} \prod_{a\geq 1} \Big(\sum_{w\in V_\sigma} \mathrm{pr}_w^*(\omega_{|w|}(a))\Big)^{m(a)} \\
&= \int_{\prod_{v\in V_\sigma} \overline{M}_{0,|v|}} \prod_{a\geq 1} \sum_{\substack{(m_w(a)|w\in V_\sigma):\\ m(a)=\sum_w m_w(a)}} \frac{m(a)!}{\prod_w m_w(a)!} \prod_{w\in V_\sigma} \mathrm{pr}_w^*(\omega_{|w|}(a))^{m_w(a)} \\
&= \int_{\prod_{v\in V_\sigma} \overline{M}_{0,|v|}} \sum_{\substack{(\mathbf{m}_w|w\in V_\sigma):\\ \mathbf{m}=\sum_w \mathbf{m}_w}} \frac{\mathbf{m}!}{\prod_w \mathbf{m}_w!} \prod_{w\in V_\sigma} \mathrm{pr}_w^*(\boldsymbol{\omega}_{|w|})^{\mathbf{m}_w} \\
&= \sum_{\substack{(\mathbf{m}_w|w\in V_\sigma):\\ \mathbf{m}=\sum_w \mathbf{m}_w}} \mathbf{m}! \prod_{v\in V_\sigma} \int_{\overline{M}_{0,|v|}} \frac{\boldsymbol{\omega}_{|v|}^{\mathbf{m}_v}}{\mathbf{m}_v!}.
\end{aligned}$$

8.3. Calculation of $\omega_n(a)$ via strata classes. For a fixed $n\geq 3$ and $a\geq 1$ consider labeled $(a+1)$–partitions

$$S: \quad \underline{n} := \{1,\dots,n\} = S_1 \amalg \dots \amalg S_{a+1}.$$

Denote by $\tau(S)$ the n–tree with $V_{\tau(S)} = \{v_1,\dots,v_{a+1}\}$ and edges connecting v_i to v_{i+1} for $i=1,\dots,a$, and unpaired flags (numbered by) S_i put at v_i. The stability condition for $\tau(S)$ and S is:

$$n_i := |S_i| \geq 2 \text{ for } i=1,a+1; \quad \geq 1 \text{ for } i=2,\dots,a. \tag{8.11}$$

In the following proof, all partitions are stable. Denote by $m(S)$ the dual cohomology class of the cycle $\varphi_{\tau(S)}(\overline{M}_{\tau(S)})$ in $\overline{M}_{0n}$.

8.3.1. Lemma. *We have*

$$\omega_n(a) = \sum_{S:\underline{n}=S_1\amalg\cdots\amalg S_{a+1}} \frac{(n_1-1)(n_{a+1}-1)n_1\dots n_{a+1}}{n(n-1)} K(n_1,\dots,n_a)\, m(S) \tag{8.12}$$

where $K(n_1,\dots,n_a)$ is defined in (8.4).

8.3.2. Notation. To state some intermediate formulas, we will need some of the notation of [KMK]. Let $T_n(a)$ be the set of n–trees with a edges. For any flag f denote by $\beta(f)$ the set of tails of the branch of f and $S(f)$ the set of their labels. Then to any set of flags T we associate the set $S(T) := \bigcup_{f\in T} S(f) \subset \{1,\dots,n\}$. If $\{S(T_1), S(T_2)\}$ is a partition of $\{1,\dots,n\}$, we use the shorthand notation D_{T_1,T_2} for $D_{S(T_1),S(T_2)}$. Let τ be an n–tree and let e be one of its edges, $\partial e = \{v_1, v_2\}$, σ_e the corresponding partition $S_1 \amalg S_2$ and D_e the corresponding divisor. Choosing flags $\{i,j\} \in F(v_1)$ and $\{j,k\} \in F(v_2)$, we have [KM] the following formula:

$$\begin{aligned} D_e m(\tau) = &- \sum_{\substack{T:\,\{i,j\}\subset T\subset F(v_1)\\ 2\le|T|\le|F(v_1)|-1}} D_{T,F(v_2)\amalg F(v_1)\setminus T}\, m(\tau) \\ &- \sum_{\substack{T:\,\{k,l\}\subset T\subset F(v_2)\\ 2\le|T|\le|F(v_2)|-1}} D_{T,F(v_1)\amalg F(v_2)\setminus T}\, m(\tau). \end{aligned}$$

8.3.3. Definition. A tree is called linear, if each vertex has at most two incident edges. An orientation of a linear tree is a labeling of its vertices by $\{1,\dots,|V(\tau)|\}$ such that v_i and v_{i+1} are connected by an edge for $i=1,\dots,|E(\tau)|$.

We denote by $LT_n(a)$ the set of stable linear n–trees with a edges modulo isomorphism. Given a geometrically oriented linear tree, we number its vertices in the positive direction.

8.3.4. Remark. The oriented linear trees in $LT_n(a)$ are in 1–1 correspondence with labeled $a+1$–partitions $S : \underline{n} := \{1,\dots,n\} = S_1 \amalg \cdots \amalg S_{a+1}$ which satisfy (8.11).

8.3.5. Tautological classes and the $\omega_n(a)$. In the proof of the Lemma, we will consider some additional classes in $H^*(\overline{M}_{g,n},\mathbb{Q})$. Let $\xi_i : \overline{M}_{g,n} \to C_n$ be the structure sections of the universal curve. Put as in [AC]:

$$\Psi_{n,i} := \xi_i^*(c_1(\omega_{C/M})) \in H^2(\overline{M}_{g,n},\mathbb{Q}). \tag{8.13}$$

Here we will need them only for $g=0$; see below for any genus.

Identify $C \to \overline{M}_{0n}$ with the forgetful morphism $p_n : \overline{M}_{0,n+1} \to \overline{M}_{0n}$. Then $\xi_i(\overline{M}_{0n})$ becomes the divisor $D_i = D_{\{i,n+1\}\{1,\dots,\hat{i},\dots,n\}}$ in $\overline{M}_{0,n+1}$ and

$$\Psi_{n+1,i} = \varphi_{D_i}^*(-D_i^2).$$

where $\varphi_{D_i}^*$ denotes the pullback on the divisor D_i. We know from [AC]:

$$\omega_{n-1}(a) = p_{n-1*}(\Psi_{n,n}^{a+1}).$$

Combining these two formulas, we obtain:

$$\omega_{n-1}(a) = p_{n-1*} \circ \varphi^*_{D_n}((-1)^{a+1} D_n^{a+2}). \tag{8.14}$$

To derive (8.14) notice that

$$\Psi_{n,i} = \sum_{n \in S \subset \{1,\dots,n\}} \frac{(n-|S|)(n-|S|-1)}{(n-1)(n-2)} D_{S,\{1,\dots,n\}\setminus S}$$

(see [KMK]) so that we have

$$\begin{aligned}
\Psi^{a+1}_{n,n} &= \Big(\sum_{n \in S \in \{1,\dots,n\}} \frac{(n-|S|)(n-|S|-1)}{(n-1)(n-2)} D_{S,\{1,\dots,n\}\setminus S} \Big)^{a+1} \\
&= \varphi^*_{D_n}(\Big(\sum_{\{n,n+1\} \subset S \in \{1,\dots,n+1\}} \frac{(n-|S|)(n-|S|-1)}{(n-1)(n-2)} D_{S,\{1,\dots,n+1\}\setminus S} \Big)^{a+1} D_n) \\
&= \varphi^*_{D_n}((-1)^{a+1} D_n^{a+2}).
\end{aligned}$$

8.3.6. A calculation. Denote by $D_nLT(k)$ the set of oriented linear $(n+1)$–trees with a edges whose monomials are divisible by D_n and whose orientation is given by calling v_{k+1} the trivalent vertex with the two tails n and $n+1$. Also take S to be the set of the flags of the other vertex v_k of the edge corresponding to D_n without the flag belonging to that edge. Then

$$D_n^k = \sum_{\tau \in D_nLT(a)} \frac{|v_1|(|v_1|-1)}{(n-1)(n-2)} \prod_{i=2}^{k-1} \frac{|v_i|-2}{\sum_{j=i}^k (|v_j|-2)} m(\tau). \tag{8.15}$$

We will prove (8.15) by induction using the following versions of (8.13). Let τ be a tree which has an edge e corresponding to D_n, then call v_2 the vertex with $F(v_2) = \{n, n+1, f_e\}$ where f_e is the flag corresponding to e.

Averaging the formula (8.13) over the set S of all flags of v_1 without the flag belonging to e, we obtain:

$$D_n m(\tau) = - \sum_{\substack{T \subset S \\ 2 \le |T| \le |F(v1)|-2}} \frac{|T|(|T|-1)}{|S|(|S|-1)} D_{T,\{n,n+1\} \amalg S\setminus T} m(\tau). \tag{8.16}$$

Fixing *one* particular flag f of S and averaging over the rest, we obtain:

$$D_n m(\tau) = - \sum_{\substack{f \in T \subset S \\ 2 \le |T| \le |F(v1)|-2}} \frac{|T|-1}{|S|-1} D_{T,\{n,n+1\} \amalg S\setminus T} m(\tau). \tag{8.17}$$

Now, for $k=1$ the formula (8.15) is clear and for $k=2$ it is a consequence of (8.17). For $k>2$, we have

$$\begin{aligned} D_n^k = D_n\, D_n^{k-1} &= D_n \sum_{\tau\in D_nLT(k-1)} \frac{|v_1|(|v_1|-1)}{(n-1)(n-2)} \prod_{i=2}^{k-2} \frac{|v_i|-2}{\sum_{j=i}^{k-1}(|v_j|-2)} m(\tau) \\ &= \sum_{\tau\in D_nLT(k-1)} \sum_{f\in T\subset S} \frac{|v_1|(|v_1|-1)}{(n-1)(n-2)} \times \\ &\qquad \prod_{i=2}^{k-2} \frac{|v_i|-2}{\sum_{j=i}^{k-1}(|v_j|-2)} \frac{|T|-1}{|v_{k-1}|-2} D_{T,\{n,n+1\}\amalg S\setminus T}\, m(\tau) \end{aligned} \tag{8.18}$$

where we have used (8.17) with the distinguished flag being the unique flag of S belonging to an edge. This guarantees that the sum will again run over linear trees. In the second sum, there is one edge inserted at the vertex v_{k-1} giving two new vertices v', v'' with $|v'|+|v''| = |v_{k-1}|+2$. Giving v', v'', the labels $k-1$ and k and labeling the old vertex v_k with $k+1$ in the second sum, we obtain the desired result (8.15).

8.3.7. Proof of the Lemma. What remains to be calculated is $p_{n-1*}\circ\varphi^*_{D_n}$ of the above formula for D_n^{a+2}. The only nonzero contributions come from trees $\tau\in D_nLT(a+2)$ with $|v_k|=3$, so that exactly one of the flags is a tail. Hence, after push–forward and pull–back, the sum will run over oriented linear trees with the induced orientation given by the image of v_i with a distinguished flag at the vertex v_{k-2}. Summing first over the possible distinguished flags amounts to multiplication by $|v_{k-1}|$. We obtain:

$$\omega_n(a) = \sum_{\text{oriented } \tau\in \mathrm{LT_n(a)}} \frac{(|v_{a+1}|-1)(|v_1|-1)}{n} \prod_{i=1}^{a+1} \frac{|v_i|-2}{\sum_{j=i}^{a+1}(|v_j|-2)+1} m(\tau)$$

which, using Remark 8.3.4, can be rewritten as a sum over partitions

$$\omega_n(a) = \sum_{S:S_1\amalg\cdots\amalg S_{a+1}} \frac{n_1 n_{a+1}}{n}(n_1-1)n_2\cdots n_a(n_{a+1}-1)\frac{1}{n-1}K(n_a+1,\dots,n_2)m(S)$$

with $n_i = |v_i|-1$ for $i=1,a+1$ and $n_i=|v_i|-2$ for $i=2,\dots,a$, which is equivalent to (8.12).

8.3.8. Remark. Instead of using (8.17) in the induction, one can successively apply (8.16). In this case, one obtains a formula for $\omega_n(a)$ involving all boundary strata. Since not necessarily linear trees cannot be handled using only partitions, the associated generating functions and recursion relations become very complicated.

8.4. Proof of Theorem 8.1.3. In view of (8.8), we have

$$\Omega_n(\mathbf{m}+\delta_a) = \int_{\overline{M}_{0n}} \prod_{b\geq 1} \frac{\omega_n(b)^{m(b)}}{m(b)!} \wedge \frac{\omega_n(a)}{m(a)+1}. \tag{8.19}$$

Instead of wedge multiplying by $\omega_n(a)$, we can integrate the product $\omega_n^{\mathbf{m}}/\mathbf{m}!$ over the cycle obtained by replacing $m(S)$ by $\varphi_{\tau(S)}(\overline{M}_{\tau(S)})$ in the r.h.s. of (8.12). The separate summands can then be calculated using (8.8) and (8.9). The result is:

$$(m(a)+1)\Omega_n(\mathbf{m}+\delta_a) = \sum_{S:\underline{n}=S_1\amalg\dots\amalg S_{a+1}} \frac{(n_1-1)(n_{a+1}-1)n_1\dots n_{a+1}}{n(n-1)} \times$$

$$K(n_1,\dots,n_a) \sum_{\mathbf{m}=\mathbf{m}_1+\dots+\mathbf{m}_{a+1}} \Omega_{n_1+1}(\mathbf{m}_1)\Omega_{n_{a+1}+1}(\mathbf{m}_{a+1}) \prod_{i=2}^{a} \Omega_{n_i+2}(\mathbf{m}_i). \tag{8.20}$$

Now, the product of Ω's vanishes unless

$$|\mathbf{m}_i| = n_i - 2 \text{ for } i = 1,\, a+1, \quad |\mathbf{m}_i| = n_i - 1 \quad \text{for } i = 2,\dots,a \tag{8.21}$$

so that $n = |\mathbf{m}+\delta_a|+3$. Hence, we can make the exterior summation over vector $(a+1)$–partitions of $\mathbf{m}$, and for a fixed $(\mathbf{m}_i)$ sum over the set of $(a+1)$–partitions of $\underline{n}$ satisfying (8.21). Since the coefficients in (8.19) depend only on (n_i) rather than (S_i), we can then replace the summation over (S_i)'s by multiplication by $\frac{n!}{n_1!\dots n_{a+1}!}$. This leads to

$$(m(a)+1)\frac{\Omega_n(\mathbf{m}+\delta_a)}{|\mathbf{m}+\delta_a|!} = (n-2) \sum_{\mathbf{m}=\mathbf{m}_1+\dots+\mathbf{m}_{a+1}} K(n_1,\dots,n_a) \prod_{i=1}^{a+1} \frac{\Omega_{|\mathbf{m}_i|+3}(\mathbf{m}_i)}{|\mathbf{m}_i|!}$$

which is equivalent to (8.5) in view of (8.8) and (8.2).

We will now quote the further results of [KMZ] without proofs:

8.5. The differential equation for a generating function. Put

$$F(x;\mathbf{s}) = \mathbf{F}(\mathbf{x};\mathbf{s_1},\mathbf{s_2},\dots) := \sum_{\mathbf{m}} \mathbf{V}(\mathbf{m})\mathbf{x}^{|\mathbf{m}|}\,\mathbf{s}^{\mathbf{m}} \in \mathbb{Q}[\mathbf{s}][[\mathbf{x}]] \tag{8.22}$$

and denote $\partial_a = \partial/\partial s_a$, $\partial_x = \partial/\partial x$. Then the recursion (8.5) is equivalent to:

8.5.1. Theorem (1.6.1 of [KMZ]). *F satisfies the following system of differential equations:*

$$\partial_a F = \partial_x\Big(\sum_{k=0}^{a} (-1)^k \frac{F^{2k+1}}{(\partial_x F)^{k+1}} \partial_{a-k} F\Big), \qquad a = 1,2,\dots \tag{8.23}$$

where we put $\partial_0 = x\,\partial_x$. It is the unique solution of this system in $1 + x\mathbb{Q}[\mathbf{s}][[\mathbf{x}]]$ with $F(x;\mathbf{0}) = 0$.

The theorem yields the above mentioned system of non–linear differential equations which can be used to obtain explicit formulas using a transformation which will be presented below.

8.6. Explicit formulas and the inversion of the generating function.

8.6.1. Notation. In this section the genus is fixed to be zero ($g \geq 0$) and g is only kept in the notation for $\overline{M}_{g,n}$ and $V_g(\mathbf{m})$, but omitted everywhere else. To state the explicit formulas, some additional classes in $H^*(\overline{M}_{g,n}, \mathbb{Q})$ need to be introduced. Recall the definition of the Ψ–classes

$$\Psi_{n,i} := \xi_i^*(c_1(\omega_{C/M})) \in H^2(\overline{M}_{g,n}, \mathbb{Q}) \tag{8.24}$$

where $\xi_i : \overline{M}_{g,n} \to C_n$ are the structure sections of the universal curve.

Following Witten [W], the integrals of top degree monomials in $\Psi_{n,i}$ are denoted

$$\langle \tau_{a_1} \dots \tau_{a_n} \rangle = \int_{\overline{M}_{g,n}} \Psi_{n,1}^{a_1} \dots \Psi_{n,n}^{a_n}. \tag{8.25}$$

For $g = 0$, they are just multinomial coefficients:

$$\langle \tau_{a_1} \dots \tau_{a_n} \rangle_{g=0} = \frac{(a_1 + \dots + a_n)!}{a_1! \dots a_n!} \tag{8.26}$$

(see e.g. [K2]). The generating series for all the correlation numbers $\langle \tau_{a_1} \dots \tau_{a_n} \rangle$ and all g was predicted by Witten [W] and later identified by Kontsevich [K1] as a "matrix Airy function".

More generally, consider the relative integrals of the type (8.25). For $k \geq l$, denote by $\pi_{k,l} : \overline{M}_{g,k} \to \overline{M}_{g,l}$ the morphism forgetting the last $k - l$ points. For any $a_1, \dots, a_p \geq 0$ define

$$\omega_n(a_1, \dots, a_p) := \pi_{n+p,n*}(\Psi_{n+p,n+1}^{a_1+1} \dots \Psi_{n+p,n+p}^{a_p+1}) \in H^{2(a_1+\dots+a_p)}(\overline{M}_{g,n}, \mathbb{Q}). \tag{8.27}$$

Notice that whenever $a_1 + \dots + a_p = \dim \overline{M}_{g,n}$, also the equation $(a_1 + 1) + \dots + (a_p + 1) = \dim \overline{M}_{g,n+p}$ holds, and therefore

$$\int_{\overline{M}_{g,n}} \omega_n(a_1, \dots, a_p) = \int_{\overline{M}_{g,n+p}} \Psi_{n+p,n+1}^{a_1+1} \dots \Psi_{n+p,n+p}^{a_p+1} = \langle \tau_0^n \tau_{a_1+1} \dots \tau_{a_p+1} \rangle. \tag{8.28}$$

8.6.2. Theorem (2.2 of [KMZ]). *For any $g, n, a_1, \dots, a_p, a_i \geq 0$, we have*

$$\omega_n(a_1) \dots \omega_n(a_p) = \sum_{k=1}^{p} \frac{(-1)^{p-k}}{k!} \sum_{\substack{\{1,\dots,p\} = S_1 \amalg \dots \amalg S_k \\ S_i \neq \emptyset}} \omega_n(\sum_{j \in S_1} a_j, \dots, \sum_{j \in S_k} a_j). \tag{8.29}$$

Equivalently, for any $\mathbf{m} \in N^\infty \setminus \{\overline{0}\}, p = \|\mathbf{m}\|$,

$$\frac{(-1)^p}{\mathbf{m}!} \boldsymbol{\omega}_n^{\mathbf{m}} = \sum_{k=1}^{p} \frac{(-1)^k}{k!} \sum_{\substack{\mathbf{m} = \mathbf{m}_1 + \dots + \mathbf{m}_k \\ \mathbf{m}_i \neq 0}} \frac{\omega_n(|\mathbf{m}_1|, \dots, |\mathbf{m}_k|)}{\mathbf{m}_1! \dots \mathbf{m}_k!}. \tag{8.30}$$

As a corollary, one obtains:

8.6.3. Corollary (2.3 of [KMZ]). *For $p = \|\mathbf{m}\|$, $3g - 3 + n = |\mathbf{m}|$:*

$$V_g(\mathbf{m}) = \frac{1}{|\mathbf{m}|!} \sum_{k=1}^{p} \frac{(-1)^{p-k}}{k!} \sum_{\substack{\mathbf{m}=\mathbf{m}_1+\dots+\mathbf{m}_k \\ \mathbf{m}_i \neq 0}} \frac{\langle \tau_0^n \tau_{|\mathbf{m}_1|+1} \dots \tau_{|\mathbf{m}_k|+1} \rangle}{\mathbf{m}_1! \dots \mathbf{m}_k!}. \tag{8.31}$$

In particular, if $g = 0$, then

$$V(\mathbf{m}) = \sum_{k=1}^{p} (-1)^{p-k} \binom{|\mathbf{m}| + k}{k} \sum_{\substack{\mathbf{m}=\mathbf{m}_1+\dots+\mathbf{m}_k \\ \mathbf{m}_i \neq 0}} \frac{1}{\prod_{k=1}^{p} (|\mathbf{m}_i| + 1)! \mathbf{m}_i!}. \tag{8.32}$$

Finally, the generalization of (1.7) which contains the announced inversion formula reads:

8.6.4. Theorem (2.4 of [KMZ]). *In the ring of formal series of one variable with coefficients in $\mathbb{Q}[\mathbf{s}] = \mathbb{Q}[\mathbf{s_1}, \mathbf{s_2}, \dots]$, we have the following inversion formula*

$$y = \sum_{|\mathbf{m}| \geq 0} V(\mathbf{m}) \frac{x^{|\mathbf{m}|+1}}{|\mathbf{m}| + 1} \mathbf{s}^{\mathbf{m}} \Longleftrightarrow \mathbf{x} = \sum_{|\mathbf{m}| \geq \mathbf{0}} \frac{\mathbf{y}^{|\mathbf{m}|+\mathbf{1}}}{(|\mathbf{m}| + \mathbf{1})!} \frac{(-\mathbf{s})^{\mathbf{m}}}{\mathbf{m}!}. \tag{8.33}$$

CHAPTER 2

Frobenius manifolds

This chapter is devoted to the study of Frobenius manifolds and their tensor product. The connection to the previous chapter is established by the very definition of the tensor product which inherently uses part of the geometry of $\overline{M}_{0n}$. In addition to this connection between Frobenius manifolds and the moduli spaces $\overline{M}_{0n}$, another part of the geometry of $\overline{M}_{0n}$, namely the higher Weil–Petersson volumes, appears in the study of formal Frobenius manifolds of dimension one.

We begin by introducing several types of Frobenius manifolds and their interrelations. These notions have been introduced by Dubrovin who basically developed the theory of Frobenius manifolds. Extended accounts can be found in the books [D], [M2] and in the articles [H] and [MM]; for further examples see [DZh].

In the second part, we will analyze the operation of forming the tensor product in each of these guises. First, we present an explicit formula for the potential function of a tensor product of formal Frobenius manifolds by applying the results of Chapter I to this particular situation. In an additional section, we review the complete description of the dimension one case following [KMZ] by reinterpreting the results about the generating functions for the higher Weil–Petersson volumes. Then, we extend the definition of forming the tensor product to the additional structures of an Euler field and an identity. Furthermore, we prove a theorem on the base–point dependence of tensor product in the case of convergent germs of Frobenius manifolds, and finally, in the case of semi–simple Frobenius manifolds, we give the special initial conditions for the tensor product of two semi–simple Frobenius manifolds in terms of the special initial conditions of the factors.

1. Formal Frobenius manifolds

1.1. Formal Frobenius manifolds. We will follow the definition from [M2]. Let k be a supercommutative $\mathbb{Q}$–algebra, $H = \oplus_{a\in A} k\partial_a$ a free ($\mathbb{Z}_2$–graded) k–module of finite rank, $g : H \otimes H \to k$ an even symmetric pairing which is non–degenerate in the sense that it induces an isomorphism $g' : H \to H^t$ where H^t is the dual module.

Denote by $K = k[[H^t]]$ the completed symmetric algebra of H^t. This means that if $\sum_a x^a \partial_a$ is a generic element of H, then K is the algebra of formal series $k[[x^a]]$. We will also regard elements of K as derivations on K with H acting via contractions. We will call the elements of H *flat*.

1.1.1. Definition. The structure of a formal Frobenius manifold on (H, g) is given by a potential $\Phi \in K$ defined up to quadratic terms which satisfies the associativity of WDVV–equations:

$$\forall a, b, c, d : \sum_{ef} \Phi_{abe} g^{ef} \Phi_{fcd} = (-1)^{\tilde{a}(\tilde{b}+\tilde{c})} \sum_{ef} \Phi_{bce} g^{ef} \Phi_{fad} \tag{1.1}$$

where $\Phi_{abc} = \partial_a \partial_b \partial_c \Phi$, g^{ij} is the inverse metric and $\tilde{a} := \widetilde{x^a} = \widetilde{\partial_a}$ is the $\mathbb{Z}_2$–degree.

From the equations (1.1) it follows that the multiplication law given by $\partial_a \circ \partial_b = \sum_c \Phi^c_{ab} \partial_c$ turns $H_K = K \otimes_k H$ into a supercommutative K–algebra.

There are two other equivalent descriptions of formal Frobenius manifolds using abstract correlation functions and $Comm_\infty$–algebras.

1.1.2. Definition. The structure of a cyclic $Comm_\infty$–algebra on (H, g) is a sequence of even polylinear maps $\circ_n : H^{\otimes n} \to H, n = 2, 3, \ldots$, called multiplications, satisfying the following conditions: We will denote $\circ_n(\gamma_1 \otimes \cdots \otimes \gamma_n)$ by $(\gamma_1, \ldots, \gamma_n)$ and call the $\circ_n | n \geq 3$ higher order multiplications.

(i) *Higher commutativity*: The multiplications $\circ_n$ are S_n–symmetric in the sense of superalgebra.

(ii) *Cyclitity*: The tensors

$$Y_{n+1} : H^{\otimes(n+1)} \to k,\ Y_{n+1}(\gamma_1 \otimes \cdots \otimes \gamma_n) := g((\gamma_1, \ldots, \gamma_n), \gamma_{n+1}) \tag{1.2}$$

are S_{n+1}–symmetric.

(iii) *Higher associativity*: $\forall m \geq 0$ and $\alpha, \beta, \gamma, \delta_a, \ldots, \delta_n$ we have:

$$\begin{aligned} &\sum_{\sigma: S_1 \amalg S_1 = \{1,\ldots,n\}} \epsilon'(\sigma)((\alpha, \beta, \delta_i | i \in S_1), \gamma, \delta_j | j \in S_2) = \\ &\sum_{\sigma: S_1 \amalg S_1 = \{1,\ldots,n\}} \epsilon''(\sigma)(\alpha, (\beta, \gamma, \delta_i | i \in S_1), \delta_j | j \in S_2). \end{aligned} \tag{1.3}$$

Here σ runs over all ordered partitions of $\{1, \ldots, m\}$ into two disjoint subsets. The signs $\epsilon'(\sigma), \epsilon''(\sigma)$ are defined as follows: fix an initial order of the arguments e.g. $\alpha, \beta, \gamma, \delta_a, \ldots, \delta_n$, then calculate the sign of the permutation induced by σ on the odd arguments in (1.3).

1.1.3. Remark. Clearly given $g, \circ_n$ and Y_{n+1} uniquely determine each other.

1.1.4. Definition. An abstract tree level system of correlation functions (ACFs) on (H, g) is a family of S_n–symmetric even polynomials

$$Y_n : H^{\otimes n} \to k,\ n \geq 3 \tag{1.4}$$

satisfying the Coherence axiom (1.5) below.

Set $\Delta = \sum \partial_a g^{ab} \partial_b$. Choose any pairwise distinct $1 \leq i, j, k, l \geq n$ and denote by $ijSkl$ any partition $S = \{S_1, S_2\}$ of $\{1, \ldots, n\}$ which separates i, j and k, l, i.e. $i, j \in S_1$ and $k, l \in S_2$. The axiom now reads:

Coherence: For any choice of i, j, k, l

$$\sum_{ijSkl}\sum_{a,b} Y_{|S_1|+1}(\bigotimes_{r\in S_1}\gamma_r\otimes\partial_a)g^{ab}Y_{|S_2|+1}(\partial_b\otimes\bigotimes_{r\in S_2}\gamma_r)$$
$$= \sum_{ikTjl}\sum_{a,b} Y_{|T_1|+1}(\bigotimes_{r\in T_1}\gamma_r\otimes\partial_a)g^{ab}Y_{|T_2|+1}(\partial_b\otimes\bigotimes_{r\in T_2}\gamma_r). \quad (1.5)$$

1.1.5. Correspondence between formal series and families of polynomials. Given a formal series $\Phi \in K$, we can expand it up to terms of order two as

$$\Phi = \sum_{n\geq 3}^{\infty}\frac{1}{n!}Y_n \quad (1.6)$$

where the $Y_n \in (H^t)^{\otimes n}$. We will consider the Y_n as even symmetric maps $H^{\otimes n} \to k$. One can check that the WDVV–equations (1.1) and the Coherence axiom (1.5) are equivalent under this identification, see e.g. [M2].

1.1.6. Theorem (III.1.5 of [M2]). *The correspondence of 1.1.5 establishes a bijection between the following structures on (H, g).*

(i) *Formal Frobenius manifolds.*
(ii) *Cyclic $Comm_\infty$–algebras.*
(iii) *Abstract correlation functions.*

1.1.7. Definition. An even element e in H_K is called an identity, if it is an identity for the multiplication $\circ$. It is called flat, if $e \in H$. In this case, we will denote e by ∂_0 and include it as a basis element.

1.1.8. Euler Operator. An even element $E \in K$ is called *conformal*, if $\mathrm{Lie}_E(g) = Dg$ for some $D \in k$. Here, we take the Lie derivative of the tensor g bilinearly extended to K w.r.t. the derivation E. In other words:

$$\forall X, Y \in K: \quad \mathrm{Lie}_E(g) := Eg(X,Y) - g([E,X],Y) - g(X,[E,Y]) = D\,g(X,Y). \quad (1.7)$$

It follows that E is the sum of infinitesimal rotation, dilation and constant shift, hence, we can write E as:

$$E = \sum_{a,b\in A} d_{ab}x^a\partial_b + \sum_{a\in A} r^a\partial_a := E_1 + E_0, \quad (1.8)$$

for some $d_{ab} \in k$. Specializing $X = \partial_a, Y = \partial_b$ we can rewrite (1.7)

$$\forall a, b: \quad \sum_c d_{ac}g_{cb} + \sum_c d_{bc}g_{ac} = Dg_{ab}. \quad (1.9)$$

In particular, we see that $[E, H] \subset H$ and that the operator

$$\mathcal{V}: H \to H: \quad \mathcal{V}(X) := [X, E] - \frac{D}{2}X \quad (1.10)$$

is skew–symmetric.

A conformal operator E is called *Euler*, if it additionally satisfies $\mathrm{Lie}_E(\circ) = d_0\circ$ for some constant d_0.

1.1.9. Quasi–homogeneity. The last condition is equivalent to the quasi–homogeneity condition (Proposition 2.2.2. of [M2])

$$E\Phi = (d_0 + D)\Phi + \text{a quadratic polynomial in flat coordinates.} \tag{1.11}$$

1.2. Operadic Correlation Functions. By identifying the index set $\overline{n} = \{1, \dots, n\}$ or more generally any finite set S with a set of markings of a S–tree, one can extend the notion of ACFs to operadic correlation functions. These are maps from $H^{\otimes S}$ which also depend on a choice of a stable S–tree τ.

$$Y(\tau) : H^{\otimes T_\tau} \to k \tag{1.12}$$

The relation of these operadic correlation functions to a given system of ACFs is provided by the following Lemma.

1.2.1. Lemma (8.4.1 of [KM]). *Starting from a system of ACFs $\{Y_n\}$, there exists a unique extension to trees, if one requires:*

(i) *For the one vertex tree with n tails ρ_n*
$$Y(\rho_n) = Y_n. \tag{1.13}$$

(ii) *Grafting together two trees τ', τ'' at the tails i, j to a tree τ corresponds to the contraction with the Casimir element:*

$$Y(\tau)(\gamma_1 \otimes \cdots \otimes \gamma_n) = \\ Y(\tau')(\gamma_1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ i}}{\Delta_a} \otimes \cdots \otimes \gamma_{n_1}) g^{ab} Y(\tau'')(\gamma_1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ j}}{\Delta_b} \otimes \cdots \otimes \gamma_{n_2}). \tag{1.14}$$

(iii) *The $Y(\tau)$ are compatible with tree isomorphisms.*

1.2.2. Remark. Given a set of ACFs $\{Y_n\}$ the correlation function of the above Lemma for a stable n–tree τ is given by the formula

$$Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) = (\bigotimes_{v \in V_\tau} Y_{F_v})(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \Delta^{\otimes |E_\tau|}). \tag{1.15}$$

We will extend this definition to the whole space $V(\Gamma_n)$ by linearity. For any element $\boldsymbol{\tau} = \sum \alpha_i \tau_i \in V(\Gamma_n)$, we set

$$Y(\boldsymbol{\tau}) := \sum \alpha_i Y(\tau_i) \tag{1.16}$$

1.2.3. Remark. To shorten the formulas, by abuse of notation, we will also denote the following function from $H^{\otimes F_\tau}$ to k by $Y(\tau)$:

$$\bigotimes_{v\in V_\tau} Y_{F_v} =: Y(\tau). \tag{1.17}$$

Which function is meant will be clear from the index set of the arguments.

1.3. Cohomological Field Theories. The structure of a formal Frobenius manifold on (H,g) is in fact equivalent to the structure of a CohFT on (H,g), see [KM],[KMK]. This is due to the fact that a (tree–level) CohFT is uniquely determined by its correlation functions.

1.3.1. Definition. A CohFT on (H,g) is given by a series of $\mathbb{S}_n$–equivariant maps:

$$I_n : H^{\otimes n} \to H^*(\overline{M}_{0n}, k), \quad n \geq 3$$

which satisfy the relations:

$$\varphi_\sigma^*(I_n(\gamma_1 \otimes \ldots \otimes \gamma_n)) = \epsilon(\sigma)(I_{n_1+1} \otimes I_{n_2+1})(\bigotimes_{j\in S_1} \gamma_j \otimes \Delta \otimes (\bigotimes_{k\in S_2} \gamma_k)) \tag{1.18}$$

where φ_σ for $\sigma = S_1 \amalg S_2$ is the inclusion map of the divisor D_σ, $\varphi_\sigma : \overline{M}_{0,|S_1|+1} \times \overline{M}_{0,|S_2|+1} \to \overline{M}_{0n}$, $\Delta = \Sigma\Delta_a \otimes \Delta_b g^{ab}$ is the Casimir element, and $\epsilon(\sigma)$ is the sign of the permutation induced on the odd arguments $\gamma_1, \ldots, \gamma_n$.

1.3.2. Equivalences of a CohFT and a system of ACFs. Given a CohFT, the associated system of ACFs is defined as follows:

$$Y_n(\gamma_1 \otimes \cdots \otimes \gamma_n) = \int_{\overline{M}_{0n}} I_n(\gamma_1 \otimes \cdots \otimes \gamma_n). \tag{1.19}$$

Given the ACFs, one can equivalently pass to the potential

$$\Phi(\gamma) := \sum_{n\geq 3} \frac{1}{n!} \int_{\overline{M}_{0n}} I_{0n}(\gamma^{\otimes n}) = \sum_{n\geq 3} \frac{1}{n!} Y_n(\gamma^{\otimes n}). \tag{1.20}$$

The reverse direction of (1.19), i.e. the reconstruction of a CohFT from its ACFs, is contained in the second reconstruction theorem of [KM]. In this context, the I_n can be recovered by extending the Y_n to a set of operadic ACFs. Then the I_n themselves can be calculated via their Poincaré duals with the help of the formula:

$$Y(\tau)(\gamma_1 \otimes \ldots \otimes \gamma_n) = \int_{\overline{M}_\tau} \varphi^*(I_n(\gamma_1 \otimes \ldots \otimes \gamma_n)). \tag{1.21}$$

The explicit calculation of the maps I_n given a potential Φ or a set of Y_n thus depends on the knowledge of the Poincaré duality as noted in [KMK] and is made possible by the results of Chapter I.

2. Germs of pointed Frobenius manifolds

2.1. Frobenius manifolds.

2.1.1. Definition. Following [M2], we define a Frobenius manifold $\mathbf{M}$ to be a quadruple $(M, \mathcal{T}_M^f, g, \Phi)$ of a (super)manifold M, an affine flat structure $\mathcal{T}_M^f$, a compatible metric g and a potential function whose tensor of third derivatives defines an associative commutative multiplication $\circ$ on each fiber of $\mathcal{T}_M$.

For the notion of supermanifolds and supergeometry in general we refer to the book [M3].

2.1.2. Definition. A *pointed Frobenius manifold* is a pair $(\mathbf{M}, m_0)$ of a Frobenius manifold $\mathbf{M}$ and a point $m_0 \in M$ called the base–point.

When considering *flat coordinates* in a neighborhood of the base–point m_0 of a pointed Frobenius manifold, we require that the coordinates of m_0 are all zero. In other words, the base–point corresponds to a choice of a zero–point in flat coordinates.

2.2. Euler field and Identity. Just as in the formal case, a Frobenius manifold may carry two additional structures; an Euler field and an identity. They are defined analogously.

2.2.1. Definition. An even vector field E on a Frobenius manifold with a flat metric g is called conformal of conformal weight D, for some constant D, if it satisfies $\mathrm{Lie}_E(g) = Dg$. A conformal field E is called *Euler*, if it additionally satisfies $\mathrm{Lie}_E(\circ) = d_0 \circ$ for some constant d_0.

2.3. From germs of pointed Frobenius manifolds to formal Frobenius manifolds. Regarding a germ of a pointed Frobenius manifold over a field k of characteristic zero, choose a flat basis of vector fields (∂_a) and set $H = \oplus_a k\partial_a$ and keep the metric g. Choose corresponding unique local flat coordinates x^a s.t. $\forall a : x^a(m_0) = 0$ as we demanded in 2.1.2. A structure of a formal Frobenius manifold on (H, g) is then given by the expansion of the potential into a power series in local flat coordinates (x^a) at m_0. Up to quadratic terms we obtain:

$$\Phi(\mathbf{x}) = \sum_{n\geq 3} \frac{1}{n!} \sum_{a_1,\dots,a_n \in \{1,\dots,n\}} x^{a_n} \cdots x^{a_1} Y_n(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) \tag{2.1}$$

where the functions Y_n are defined via

$$Y_n(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) := \partial_{a_1} \cdots \partial_{a_n} \Phi|_0 \tag{2.2}$$

Φ obviously obeys the WDVV–equations.

Furthermore, in the presence of an Euler field or a flat identity writing E and $e = \partial_0$ in flat coordinates defines the same structures in the formal situation.

We stress again that we are dealing with pointed Frobenius manifolds. Due to this a zero in flat coordinates has been fixed and $\{Y_n\}, E$ and e are uniquely defined.

On the other hand, the functions in (2.2) are dependent on the choice of the base–point. Choosing a different base–point $\widehat{m}_0$ with x–coordinates $x^a(\widehat{m}_0) = x_0^a$ in the domain of convergence of Φ yields the new standard flat coordinates $\widehat{x}^a = x^a - x_0^a$. The corresponding functions Y transform via:

$$\begin{aligned}\widehat{Y}_n(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) &:= \partial_{a_1} \ldots \partial_{a_n} \Phi|_{x_0} \\ &= \sum_{N \geq 0} \frac{1}{N!} \sum_{(b_1,\ldots,b_N):b_i \in A} x_0^{b_N} \cdots x_0^{b_1} Y_{n+N}(\partial_{b_1} \otimes \cdots \otimes \partial_{b_N} \otimes \partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) \\ &= \sum_{N \geq 0} \frac{1}{N!} \sum_{(b_1,\ldots,b_N):b_i \in A} \epsilon(b|a)\, x_0^{b_N} \cdots x_0^{b_1} Y_{n+N}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \partial_{b_1} \otimes \cdots \otimes \partial_{b_N})\end{aligned} \tag{2.3}$$

where $\epsilon(b|a)$ is a shorthand notation for $\epsilon(\partial_{b_1} \cdots \partial_{b_N} | \partial_{a_1} \ldots \partial_{a_n})$ which we define as the superalgebra sign acquired by permuting $\partial_{b_1}, \ldots, \partial_{b_N}$ past the $\partial_{a_1}, \ldots, \partial_{a_n}$:

$$\partial_{b_1} \cdots \partial_{b_N} \partial_{a_1} \cdots \partial_{a_n} = \epsilon(b|a)\, \partial_{a_1} \cdots \partial_{a_n} \partial_{b_1} \cdots \partial_{b_N}. \tag{2.4}$$

2.4. From convergent formal Frobenius manifolds to germs of pointed Frobenius manifolds. Starting with any formal Frobenius manifold (H, g) with a potential Φ, we can produce a germ of a manifold with a flat structure by identifying the x^a as coordinate functions around some point m_0, choosing H as the space of flat fields and considering g as the metric. To get a Frobenius manifold, however, we need that the formal potential Φ has some nonempty domain of convergence. If

$$\Phi(\gamma) = \sum_{n \geq 3} \frac{1}{n!} Y_n(\gamma^{\otimes n}) \tag{2.5}$$

with $\gamma = \sum x^a \Delta_a$ is convergent, we can pass to a germ of a pointed Frobenius manifold. If necessary, we can, in this situation, even move the base–point as indicated above.

3. Semi–simple Frobenius manifolds

3.1. Semi–simple Frobenius manifolds and Schlesinger equations. We will briefly recall the main notions of semi–simple Frobenius manifolds as explained in [M2]. For other versions see [D] or [H]. A Frobenius manifold of dimension n is called semi–simple (respectively split semi–simple), if an isomorphism of the sheaves of $\mathcal{O}_M$–algebras

$$(\mathcal{T}_M, \circ) \simeq (\mathcal{O}_M^n, \text{ componentwise multiplication}) \tag{3.1}$$

exists everywhere locally (respectively globally).

If a Frobenius manifold M is semi–simple, one can find so–called canonical coordinates u_i —unique up to constant shifts and renumbering— s. t. the metric and the three–tensor A defining the multiplication become particularly simple. Let $e_i = \frac{\partial}{\partial u_i}, \nu_i = \mathrm{d}u_i$, then

$$g = \sum_i \eta_i(\nu_i^2), \tag{3.2}$$

$$A = \sum_i \eta_i(\nu_i^3). \tag{3.3}$$

If in addition an Euler field exists, then it has the form $E = \sum (u^i + c^i)e_i$. In this situation, we will normalize the coordinates in such a way that

$$E = \sum u^i e_i. \tag{3.4}$$

This normalization fixes the ambiguity in the coordinates u^i and renders them unique up to the $\mathbb{S}_n$–action.

3.2. Definition. In the above situation, we will call a point $m \in M$ tame, if it satisfies $u_i(m) \neq u_j(m)$ for all $i \neq j$. In other words, the point m is tame, if the spectrum of the operator $E\circ$ on $\mathcal{T}_M$ is simple.

3.3. The extended structure connection and the second structure connection. As exhibited in [M2], the structure connection ∇_λ which is defined by

$$\nabla_{\lambda,X}(Y) = \nabla_{0,X}(Y) + \lambda X \circ Y \tag{3.5}$$

where ∇_0 is the Levi–Civita connection for g, gives rise to two extended structure connections:

$$\widehat{\nabla} \text{ on the sheaf } \mathrm{pr}_M^*(\mathcal{T}_M) \text{ on } \widehat{M} = M \times \mathbb{P}^1_\lambda \setminus \{0, 1, \infty\} \tag{3.6}$$

and

$$\check{\nabla} \text{ on the sheaf } \check{\mathcal{T}} = \mathrm{pr}_M^*(\mathcal{T}_M \big|_{\check{M}}) \text{ on } \check{M} = \bigcup_\lambda (M_\lambda \times \{\lambda\}) \subset M \times \mathbb{P}^1_\lambda \tag{3.7}$$

where $M_\lambda \subset M$ is the open subset defined by $\forall i : u^i \neq \lambda$.

Since it can be shown that the poles of $\check{\nabla}$ are all of order 1, this connection can be regarded as an isomonodromic deformation of a meromorphic connection on $\mathbb{P}^1$. These deformations are governed by the Schlesinger differential equations [Sch], [Mal], thus providing a link between Frobenius manifolds and solutions of the Schlesinger equations; the details can be found in [M2] and [MM].

3.4. Definition. A *solution to Schlesinger's equation* consists of any datum $(M,(u^i),T,(A_i))$ where M is a complex manifold of dimension $m \geq 2$, the tuple $(u^1,\dots,u^m)$ is a system of holomorphic functions on M with the properties that for any $i \neq j, x \in M$, we have $u^i(x) \neq u^j(x)$, and $\mathrm{d}u^i$ freely generate Ω^1_M, T a finite dimensional complex vector space and the $A_j : M \to \mathrm{End}(T)$, $j = 1,\dots,m$ are a family of holomorphic matrix functions satisfying

$$\forall j: \quad \mathrm{d}A_j = \sum [A_i, A_j] \frac{\mathrm{d}(u^i - u^j)}{u^i - u^j}. \tag{3.8}$$

Summing the above equation over all j shows that $\sum_j A_j$ is a constant matrix which will be called $\mathcal{W}$.

The main definition w.r.t. to the theory of Frobenius manifolds is:

3.5. Definition. A solution to Schlesinger's equations is called special, if $\dim(T) = m = \dim M$; T is endowed with a complex non–degenerate quadratic form $g; \mathcal{W} = -\mathcal{V} - \frac{1}{2}\mathrm{Id}$ where $\mathcal{V} \in \mathrm{End}(T)$ is a skew–symmetric operator, w.r.t. g, and

$$\forall j: \quad A_j = -(\mathcal{V} + \frac{1}{2}\mathrm{Id})P_j, \tag{3.9}$$

where $P_j : M \to \mathrm{End}(T)$ is a family of matrix functions whose values at any point of M constitute a complete system of orthogonal projectors of rank one w.r.t. g.

A solution is called *strictly special*, if the operators

$$A_j^{(t)} := A_j + tP_j \tag{3.10}$$

also satisfy Schlesinger's equations for any $t \in \mathbb{C}$.

3.6. Identity. Call a vector e in T an identity of weight D, if

$$\mathcal{V}(e) = (1 - \frac{D}{2})e \quad \text{and} \tag{3.11}$$

$$e_j := P_j(e) \quad \text{do not vanish at any point of } M. \tag{3.12}$$

3.7. Theorem (2.6.1 of [MM]). Let $(M, (u^i), T, (A_i))$ be a strictly special solution and e an identity of weight D, then these data come from a unique structure of semi–simple split Frobenius manifold M with an identity ($d_0 = 1$) and an Euler field via

$$T = \Gamma(M, \mathcal{T}_M^f) \tag{3.13}$$

$$(u^i): \quad \text{the canonical coordinates} \tag{3.14}$$

$$A_j(e_i) = 0 \quad \text{for } i \neq j \tag{3.15}$$

$$A_i(e_i) = -\frac{1}{2}e_i + \sum_{j:j\neq i}(u^j - u^i)\frac{\eta_{ij}}{\eta_i}. \tag{3.16}$$

The operator $\mathcal{V}$ is given by:

$$\mathcal{V}(X) = \nabla_{0,X}(E) - \frac{D}{2}X. \tag{3.17}$$

Here, the manifold M only has tame points which means that by definition $u^i(m) \neq u^j(m), \forall i \neq j, m \in M$. M should be regarded as a splitting cover of the subspace of tame points of a given Frobenius manifold.

3.8. Special initial conditions. Fixing a base–point in a solution to Schlesinger's equations and taking the coordinates e_i for T call a family of matrices $A_1^0, \dots A_m^0 \in \mathrm{End}(T)$ *special initial conditions*, if there exists a diagonal metric g and a skew–symmetric operator $\mathcal{V}$ s. t. $A_j^0 = -(\mathcal{V} + \frac{1}{2}\mathrm{Id})P_j$, where P_j is the projector onto $\mathbb{C}e_j$.

In the case of semi–simple Frobenius manifolds with an Euler field and a flat identity, the special initial conditions are given by the value of the structures listed in 3.7 at a fixed tame point $m_0 \in M$ with coordinates (u_0^i); more precisely, the metric is given by the $\eta_i(m_0)$ and the operator $\mathcal{V}$ by the matrix $(v_{ij})_{ij}$ defined by $(\nabla_{0,e_i}(E) - \frac{D}{2}(e_i))(m_0) = (\sum_j v_{ij}e_j)(m_0)$.

4. The tensor product of formal Frobenius manifolds

The tensor product of two formal Frobenius manifolds is defined via the corresponding sets of CohFTs. Looking at all CohFTs together with the operation of forming the tensor product, we can regard them naively as an infinite dimensional algebraic variety with a structure of a semigroup on it. In this setting, a natural question in view of Theorem 1.1.6 is the behaviour of the potential as a function on this moduli space. In particular, we would like to understand how to express the potential function $\Phi_{\mathcal{A}'\otimes\mathcal{A}''}$ associated to the tensor product of two CohFTs $\mathcal{A}' = (H', g', I')$ and $\mathcal{A}'' = (H'', g'', I'')$ in terms of the potential functions $\Phi_{\mathcal{A}'}$ and $\Phi_{\mathcal{A}''}$.

Due to the fact that not the functions themselves, but rather the CohFTs they represent are tensored, the operation of forming the tensor product incorporates a part of the geometry of the moduli spaces $\overline{M}_{0n}$ which will be explained below.

Using our results of Chapter I, we derive an explicit formula expressing the potential of a tensor product via the potentials of the factors answering the above question.

4.1. The diagonal of $\overline{M}_{0n} \times \overline{M}_{0n}$. Denote the class of the diagonal in $A^{n-3}(\overline{M}_{0n} \times \overline{M}_{0n})$ by $\Delta_{\overline{M}_{0n}}$ and choose an inverse image of this class in $V(\Gamma_n) \otimes V(\Gamma_n)$ under the map *mon* of Section I.3.3.1 which we —by abuse of notation— also denote by $\Delta_{\overline{M}_{0n}}$. We can choose the inverse image in such a way that it has an expansion:

$$\Delta_{\overline{M}_{0n}} = \sum_{\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathcal{B}_n} \boldsymbol{\sigma} g^{\boldsymbol{\sigma}\boldsymbol{\tau}} \otimes \boldsymbol{\tau} \in V(\Gamma_n) \otimes V(\Gamma_n) \tag{4.1}$$

where the $\boldsymbol{\tau} \in \mathcal{B}_n$ are homogeneous elements of $V(\Gamma_n)$, $\{mon(\boldsymbol{\tau})|\boldsymbol{\tau} \in \mathcal{B}_n\}$ is a basis of $A^*(\overline{M}_{0n})$ and $g_{\boldsymbol{\sigma}\boldsymbol{\tau}} = \int_{\overline{M}_{0n}} mon(\boldsymbol{\sigma}) \cup mon(\boldsymbol{\tau})$. Notice that

$$g^{\boldsymbol{\sigma}\boldsymbol{\tau}} = 0 \ \text{ unless } \ \boldsymbol{\sigma} \otimes \boldsymbol{\tau} \in V(\Gamma_{n,e}) \otimes V(\Gamma_{n,n-3-e}). \tag{4.2}$$

We can and will use the basis $\mathcal{B}_n$ of Chapter I for this purpose.

4.1.1. Remark. Note that the $\boldsymbol{\tau}$ need not be trees. However, they can be chosen as a linear combination of trees of the same degree, and furthermore generalizing the spaces $V(\Gamma)$ to trees with multiplicities $V(\Gamma^{mult})$ a basis of $A^*(\overline{M}_{0n})$ can even be chosen in basis elements of $V(\Gamma^{mult})$, as was shown in Chapter I.

4.1.2. Tensor product for CohFT. In the language of CohFT, the tensor product of $(H', g', \{I'_n\})$ and $(H'', g'', \{I''_n\})$ is given by the tensor product CohFT on $H' \otimes H''$ which is naturally defined via the cup product in $H^*(\overline{M}_{0n}, k)$:

$$\begin{aligned}(I'_n \otimes I''_n)(\gamma'_1 \otimes \gamma''_1 \otimes \cdots \otimes \gamma'_n \otimes \gamma''_n) &:= \\ \epsilon(\gamma', \gamma'') I'_n(\gamma'_1 \otimes \cdots \otimes \gamma'_n) &\wedge I''_n(\gamma''_1 \otimes \cdots \otimes \gamma''_n)\end{aligned} \tag{4.3}$$

where $\epsilon(\gamma', \gamma'')$ is the superalgebra sign.

Using 1.3.2 and Theorem 1.1.6, one can formally transfer this definition of the tensor product onto any of the other structures $(Y_n, Y(\tau), \Phi, \circ_n, \circ(\tau))$.

4.2. The naive tensor product. For any two sets of ACFs $\{Y'_n\}$ and $\{Y''_n\}$ and any element $\boldsymbol{\tau} \otimes \boldsymbol{\sigma} \in V(\Gamma_n) \otimes V(\Gamma_n)$ set:

$$(Y' \otimes Y'')(\boldsymbol{\tau} \otimes \boldsymbol{\sigma})((\gamma'_1 \otimes \gamma''_1) \otimes \cdots \otimes (\gamma'_n \otimes \gamma''_n)) = \\ \epsilon(\gamma', \gamma'') Y'(\boldsymbol{\tau})(\gamma'_1 \otimes \cdots \otimes \gamma'_n) Y''(\boldsymbol{\sigma})(\gamma''_1 \otimes \cdots \otimes \gamma''_n). \quad (4.4)$$

4.3. Tensor product for ACFs. Translating (4.3) into the language of ACFs, we obtain the following formula for the tensor product of Y'_n and Y''_n:

$$(Y'_n \otimes Y''_n)(\gamma'_1 \otimes \gamma''_1 \otimes \cdots \otimes \gamma'_n \otimes \gamma''_n) = \\ \epsilon(\gamma', \gamma'') \int_{\overline{M}_{0n}} I'_n(\gamma'_1 \otimes \cdots \otimes \gamma'_n) \wedge I''_n(\gamma''_1 \otimes \cdots \otimes \gamma''_n). \quad (4.5)$$

4.3.1. Lemma. The tensor product for two systems of ACFs $(T', \Delta', \{Y'_n\})$ and $(T'', \Delta'', \{Y''_n\})$ is the system of ACFs $(T' \otimes T'', \Delta' \otimes \Delta'', \{Y_n\})$ where

$$Y_n((\gamma'_1 \otimes \gamma''_1) \otimes \cdots \otimes (\gamma'_n \otimes \gamma''_n)) := \\ \epsilon(\gamma', \gamma'')(Y' \otimes Y'')(\Delta_{\overline{M}_{0n}})((\gamma'_1 \otimes \gamma''_1) \otimes \cdots \otimes (\gamma'_n \otimes \gamma''_n)). \quad (4.6)$$

4.3.2. Definition. Given two formal Frobenius manifolds (H', g', Φ') and (H'', g'', Φ''), let $\{Y'_n\}$ and $\{Y''_n\}$ be the corresponding ACFs. The tensor product (H, g, Φ) of (H', g', Φ') and (H'', g'', Φ'') is defined to be $(H' \otimes H'', g' \otimes g'', \Phi)$ where the potential Φ is given by:

$$\Phi(\gamma) = \sum_{n \geq 3} \frac{1}{n!}(Y' \otimes Y'')(\Delta_{\overline{M}_{0n}})(\gamma^{\otimes n}). \quad (4.7)$$

As in 1.1.5, to make sense of (4.7) one should expand $\gamma = \sum x^{a', a''} \partial_{a'a''}$ in terms of the tensor product basis $(\partial_{a'a''} := \partial'_{a'} \otimes \partial''_{a''})$ of the two basis $\{\partial'_{a'}\}$ and $\{\partial''_{a''}\}$ and the dual coordinates $x^{a'a''}$ for this basis.

The results of Chapter I now allow us to calculate (4.7) explicitly:

4.3.3. Corollary. *The explicit formal Frobenius manifold structure for the tensor product of two formal Frobenius manifolds (H', g', Φ') and (H'', g'', Φ'') is given by potential Φ on $(H' \otimes H'', g' \otimes g'')$ corresponding to the ACFs:*

$$(Y_n)(\gamma'_1 \otimes \gamma''_1 \otimes \cdots \otimes \gamma'_n \otimes \gamma''_n) = \\ \epsilon(\gamma', \gamma'') \sum_{\mu, \nu \in \mathcal{B}_n} Y'(\check{\mu})(\gamma'_1 \otimes \cdots \otimes \gamma'_n) m_{\mu\nu} Y''(\check{\nu})(\gamma''_1 \otimes \cdots \otimes \gamma''_n), \quad (4.8)$$

where $\mathcal{B}_n$ is the basis of Chapter I and the $\check{\mu}, \check{\nu}$ are the duals to μ, ν.

4.4. Examples.

4.4.1. Higher order correlation functions. Using the calculations of the Section I.7.4 and omitting the supersign, we obtain the following formulas for the tensor product of the first higher order correlation functions and higher order multiplications of (H', g', Y'_n) and (H'', g'', Y''_n).

Let $\sum_{a'b'} \Delta'_{a'} g'^{a'b'} \Delta'_{b'}$ and $\sum_{a''b''} \Delta''_{a''} g''^{a''b''} \Delta''_{b''}$ the Casimir elements for g and g'.

n=3

$$(Y'_3 \otimes Y''_3)(\gamma'_1 \otimes \gamma''_1 \otimes \gamma'_2 \otimes \gamma''_2 \otimes \gamma'_3 \otimes \gamma''_3) = Y'_3(\gamma'_1 \otimes \gamma'_2 \otimes \gamma'_3) Y'_3(\gamma''_1 \otimes \gamma''_2 \otimes \gamma''_3) \tag{4.9}$$

n=4

$$\begin{aligned}(Y'_4 \otimes Y''_4)&(\gamma'_1 \otimes \gamma''_1 \otimes \cdots \otimes \gamma'_4 \otimes \gamma''_4) = \\ &Y'_4(\gamma'_1 \otimes \cdots \otimes \gamma'_4) \sum_{a'',b''} Y''_3(\gamma''_1 \otimes \gamma''_2 \otimes \Delta''_{a''}) g''^{a''b''} Y''_3(\Delta''_{b''} \otimes \gamma''_3 \otimes \gamma''_4) + \\ &\quad \sum_{a',b'} Y'_3(\gamma'_1 \otimes \gamma'_2 \otimes \Delta'_{a'}) g'^{a'b'} Y'_3(\Delta'_{b'} \otimes \gamma'_3 \otimes \gamma'_4) Y''_4(\gamma''_1 \otimes \cdots \otimes \gamma''_4)\end{aligned} \tag{4.10}$$

n=5

$$\begin{aligned}&(Y'_5 \otimes Y''_5)(\gamma'_1 \otimes \gamma''_1 \otimes \cdots \otimes \gamma'_5 \otimes \gamma''_5) = \\ &\sum_{a'',b'',c'',d''} Y''_3(\gamma''_1 \otimes \gamma''_2 \otimes \Delta''_{a''}) g''^{a''b''} Y''_3(\Delta''_{b''} \otimes \gamma''_3 \otimes \Delta''_{c''}) g''^{c''d''} Y''_3(\Delta''_{d''} \otimes \gamma''_4 \otimes \gamma''_5) \\ &\quad \times Y'_5(\gamma'_1 \otimes \cdots \otimes \gamma'_5) \\ &- \sum_{l \in \{1,2,3,4\}} \sum_{\substack{a',b' \\ a'',b''}} Y'_4\Big(\bigotimes_{i \in \{1,2,3,4\} \setminus \{l\}} \gamma'_i \otimes \Delta'_{a'}\Big) g'^{a'b'} Y'_3(\Delta'_{b'} \otimes \gamma'_l \otimes \gamma'_5) \\ &\quad \times Y''_4\Big(\bigotimes_{i \in \{1,2,3,4\} \setminus \{l\}} \gamma''_i \otimes \Delta''_{a''}\Big) g''^{a''b''} Y''_3(\Delta''_{b''} \otimes \gamma''_l \otimes \gamma''_5) \\ &+ \sum_{\{1,2\} \subseteq I \subset \{1,2,3,4\}} \sum_{a',b'} Y'_{|I|+1}\Big(\bigotimes_{i \in I} \gamma'_i \otimes \Delta'_{a'}\Big) g'^{a'b'} Y'_{6-|I|}\Big(\Delta'_{b'} \bigotimes_{j \in \{1,2,3,4\} \setminus I} \gamma'_j \otimes \gamma'_5\Big) \\ &\quad \times \sum_{\{1,2\} \subseteq J \subset \{1,2,3,4\}} \sum_{a'',b''} Y''_{|J|+1}\Big(\bigotimes_{i \in J} \gamma''_i \otimes \Delta''_{a''}\Big) g'^{a''b''} Y''_{6-|I|}\Big(\Delta''_{b''} \bigotimes_{j \in \{1,2,3,4\} \setminus J} \gamma''_j \otimes \gamma''_5\Big) \\ &+ \sum_{a',b',c',d'} Y'_3(\gamma'_1 \otimes \gamma'_2 \otimes \Delta'_{a'}) g'^{a'b'} Y'_3(\Delta'_{b'} \otimes \gamma'_3 \otimes \Delta'_{c'}) g'^{c'd'} Y'_3(\Delta'_{d'} \otimes \gamma'_4 \otimes \gamma'_5) \\ &\quad \times Y''_5(\gamma''_1 \otimes \cdots \otimes \gamma''_5).\end{aligned} \tag{4.11}$$

4.4.2. Higher order multiplications. By applying equation (1.2), again using the notation $(\gamma_1, \ldots, \gamma_n)$ for $\circ_n(\gamma_1 \otimes \cdots \otimes \gamma_n)$, we find:

n=2

$$(\gamma'_1 \otimes \gamma''_1, \gamma'_2 \otimes \gamma''_2) = (\gamma'_1, \gamma'_2) \otimes (\gamma''_1, \gamma''_2) \tag{4.12}$$

n=3

$$(\gamma_1' \otimes \gamma_1'', \gamma_2' \otimes \gamma_2'', \gamma_3' \otimes \gamma_3'') = \\ (\gamma_1', \gamma_2', \gamma_3') \otimes ((\gamma_1'', \gamma_2''), \gamma_3'') + ((\gamma_1', \gamma_2'), \gamma_3') \otimes (\gamma_1'', \gamma_2'', \gamma_3'') \quad (4.13)$$

n=4

$$(\gamma_1' \otimes \gamma_1'', \dots, \gamma_4' \otimes \gamma_4'') = \\ (\gamma_1', \dots \gamma_4') \otimes (((\gamma_1'', \gamma_2''), \gamma_3''), \gamma_4'') + (((\gamma_1', \gamma_2'), \gamma_3'), \gamma_4') \otimes (\gamma_1'', \dots \gamma_4'') \\ - \sum_{\{i,j,k\} \amalg \{l\} = \{1,2,3,4\}} ((\gamma_i', \gamma_j', \gamma_k'), \gamma_l') \otimes ((\gamma_i'', \gamma_j'', \gamma_k''), \gamma_l'') \\ + \sum_{\{1,2\} \subseteq I \subset \{1,2,3,4\}} ((\gamma_I'), \gamma'_{\{1,2,3,4\} \setminus I}) \otimes \sum_{\{1,2\} \subseteq J \subset \{1,2,3,4\}} ((\gamma_J''), \gamma''_{\{1,2,3,4\} \setminus J}), \quad (4.14)$$

where in the last expression we have used the abbreviation (γ_I) to denote $\circ_{|I|}(\otimes_{i \in I} \gamma_i)$.

5. Potential of the invertible Cohomological Field Theories

We now make a digression to the space of formal Frobenius manifolds or equivalently CohFTs of dimension one and the operation of forming the tensor product in these theories following [KMZ]. In this special situation, there are even more results with respect to the general problem of expressing the potential function $\Phi_{\mathcal{A}'\otimes\mathcal{A}''}$ associated to the tensor product of two CohFTs $\mathcal{A}' = (H', g', I')$ and $\mathcal{A}'' = (H'', g'', I'')$ in terms of the potential functions $\Phi_{\mathcal{A}'}$ and $\Phi_{\mathcal{A}''}$. This is achieved by using a connection to the theorems about the higher Weil–Petersson volumes of Chapter I. In fact, see [KMK], the genus zero generating function for higher WP–volumes is the third derivative of the potential of a generic invertible CohFT of dimension one written in coordinates additive with respect to the tensor multiplication.

5.1. The moduli space of one–dimensional CohFTs. We will consider CohFT structures on one–dimensional spaces. In [KMK] such a theory was shown to be invertible with respect to the tensor product, if the map I_3 from $H^{\otimes 3} \cong k$ to $H^*(\overline{M}_{03}, k) \cong k$ is an isomorphism. A one–dimensional theory is called normalized, if a basis of length one is fixed, $H = k\Delta_0$, $g(\Delta_0, \Delta_0) = 1$, and $I_3(\Delta_0 \otimes \Delta_0 \otimes \Delta_0) = 1$. Equivalently, $I_n(\Delta_0^{\otimes n}) = 1_n +$ terms of dimension > 0 for all $n \geq 3$, where $1_n \in H^0(\overline{M}_{0n}, k)$ is the fundamental class. In this case, the potential function has the form

$$\Phi_{\mathcal{A}}(x) = \sum_{n=3}^{\infty} C_n \frac{x^n}{n!} \tag{5.1}$$

where $C_3 = 1$ and the other coefficients are arbitrary, since the associativity equations are empty in this case and they are the only restriction by virtue of Theorem 1.1.6. Regarding the space $\mathbf{CohFT_1}(k)$ of all normalized and invertible 1–dimensional CohFTs, we thus see that it is canonically isomorphic to $\frac{1}{6}x^3 + x^4\, k[[x]]$ and has canonical coordinates C_n $(n \geq 4)$. The goal is now to describe the tensor product in terms of these coordinates which encode the potential.

In [KMK] where these 1–dimensional CohFT structures were previously studied, a different set of coordinates was given which behaves nicely with respect to forming the tensor product.

Namely, for each sequence $s_1, s_2, \ldots; s_i \in k$ there is an element $\mathcal{A}(\mathbf{s}) \in \mathbf{CohFT_1}(k)$ given by

$$I_n(\Delta_0^{\otimes n}) = \omega_n[s_1, s_2, \ldots] := \exp\Big(\sum_{a=1}^{\infty} s_a \omega_n(a)\Big) \qquad (n \geq 3). \tag{5.2}$$

In [KMK] it was shown that the map $\mathbf{s} \mapsto \mathcal{A}(\mathbf{s})$ yields a bijection between $k^{\mathbb{N}}$ and $\mathbf{CohFT_1}(k)$ and that $\mathcal{A}(\mathbf{s}') \otimes \mathcal{A}(\mathbf{s}'') \cong \mathcal{A}(\mathbf{s}' + \mathbf{s}'')$, i.e.:

5.2. Theorem (3.2.1 of [KMK]). *The parameters $(s_1, s_2, \dots)$ form a coordinate system on the space of normalized 1–dimensional CohFTs. The tensor product becomes addition in these coordinates.*

Let $\Phi(x; \mathbf{s})$ denote the potential associated to the theory (5.2). The third derivative of the potential $\Phi(x; \mathbf{s})$ associated to the theory (5.2) is just the generating function for higher WP–volumes, connecting the present consideration to what has been presented in Section I.8.

Looking at the definition for the potential we see:

$$\Phi(x; s_1, s_2, \dots) = \sum_{n=3}^{\infty} \frac{x^n}{n!} \int_{\overline{M}_{0n}} \sum_{\sum am(a)=n-3} \prod_{a\geq 1} \omega_n(a)^{m(a)} \frac{s_a^{m(a)}}{m(a)!},$$

and the third derivative of this function $\Phi(x; s_1, s_2, \dots)$ is obviously the function $F(x; \mathbf{s}) = \sum_{\mathbf{m}} V(\mathbf{m}) x^{|\mathbf{m}|} \mathbf{s}^{\mathbf{m}}$ defined in II.8.5. This observation can be used to describe both the tensor product and the coordinates on the space of invertible 1–dimensional CohFTs explicitly.

5.3. Theorem (3.4.2 of [KMZ]). *Define the bijections*

$$\mathbf{CohFT_1}(k) \;\longleftrightarrow\; \frac{x^3}{6} + x^4\, k[[x]] \;\longleftrightarrow\; 1 + \eta\, k[[\eta]]\,, \tag{5.3}$$

where the first map assigns to a theory $\mathcal{A}$ its potential $\Phi_{\mathcal{A}}(x)$ and the second map is defined by

$$\Phi(x) \;\leftrightarrow\; U(\eta) = \int_0^{\infty} e^{-\Phi''(\eta x)/\eta}\, dx \tag{5.4}$$

or alternatively by assigning to $\Phi(x) = \frac{1}{6}x^3 + \dots$ the power series $U(\eta) = \sum_{n=0}^{\infty} B_n \eta^n$ where $x = \sum B_n \frac{y^{n+1}}{(n+1)!} = y + \cdots$ is the inverse power series of $y = \Phi''(x) = x + \cdots$. Then the tensor product of 1–dimensional CohFTs corresponds to multiplication in $1 + \eta k[[\eta]]$: $U_{\mathcal{A}'\otimes\mathcal{A}''}(\eta) = U_{\mathcal{A}'}(\eta)\, U_{\mathcal{A}''}(\eta)$. The coefficients of $-\log U_{\mathcal{A}}(\eta)$ are the canonical coordinates of $\mathcal{A}$.

5.4. Explicit formulas. The above theorem can be used to give explicit formulas for the coefficients of $U(\eta)$ in terms of the coefficients of $\Phi(x)$. Substituting (5.1) (with $C_3 = 1$) into (5.4), expanding and integrating term by term yields:

$$B_n = \sum_{\substack{n_4, n_5, \ldots \geq 0 \\ n_4 + 2n_5 + \cdots = n}} \frac{(2n_4 + 3n_5 + \cdots)!}{2!^{n_4}\, 3!^{n_5} \cdots n_4!\, n_5! \cdots} (-C_4)^{n_4} (-C_5)^{n_5} \cdots$$

Applying the same argument to the inverse power series yields the reciprocal formula:

$$C_n = \sum_{\substack{n_1, n_2, \ldots \geq 0 \\ n_4 + 2n_5 + \cdots = n-3}} \frac{(2n_1 + 3n_2 + \cdots)!}{2!^{n_1}\, 3!^{n_2} \cdots n_1!\, n_2! \cdots} (-B_1)^{n_1} (-B_2)^{n_2} \cdots.$$

The explicit law for the tensor product of two normalized invertible CohFTs in terms of the coefficients of their potential functions can be derived by combining these formulas with the identity $U_{\mathcal{A}'\otimes\mathcal{A}''}(\eta) = U_{\mathcal{A}'}(\eta)\, U_{\mathcal{A}''}(\eta)$:

$$\begin{aligned}
C_4 &= C_4' + C_4'' \,, \\
C_5 &= C_5' + 5C_4'C_4'' + C_5'' \,, \\
C_6 &= C_6' + (8\, {C_4'}^2 + C_5')\, C_4'' + C_4'\, (8\, {C_4''}^2 + C_5'') + C_6'' \,, \\
C_7 &= C_7' + (35\, C_4'\, C_5' + 14\, C_6')\, C_4'' + (61\, {C_4'}^2\, {C_4''}^2 + 33\, {C_4'}^2 C_5'' + 33\, C_5'\, {C_4''}^2 \\
&\qquad + 19\, C_5'\, C_5'') + C_4'\, (35\, C_4''\, C_5'' + 14\, C_6'') + C_7'' \,, \qquad \ldots
\end{aligned}$$

Finally, the values of the genus zero Weil–Petersson volumes $V(\mathbf{m})$ can be calculated numerically alternatively from the recursion relation (II.8.5), the differential equation (II.8.23), the closed formula (II.8.32) or the generating function formula (II.8.33).

The generating function (II.8.22) up to $|\mathbf{m}| = 5$ reads:

$$\begin{aligned}
F(x, \mathbf{s}) = 1 \,+\, & s_1\, x \,+\, \big(5\, \frac{s_1^2}{2} + s_1\big)\, \frac{x^2}{2} \,+\, \big(61\, \frac{s_1^3}{6} + 9\, s_1 s_2 + s_3\big)\, \frac{x^3}{6} \\
+\, & \big(1379\, \frac{s_1^4}{24} + 161\, \frac{s_1^2 s_2}{2} + 14\, s_1 s_3 + 19\, \frac{s_2^2}{2} + s_4\big)\, \frac{x^4}{24} \\
+\, & \big(49946\, \frac{s_1^5}{120} + 4822\, \frac{s_1^3 s_2}{6} + 344\, \frac{s_1^2 s_3}{2} + 470\, \frac{s_1 s_3^2}{2} + 20\, s_1 s_4 + 34\, s_2 s_3 + s_5\big)\, \frac{x^5}{120} \\
+\, & \mathrm{O}(x^6) \,.
\end{aligned}$$

The coefficient $\int_{\overline{M}_{0,n}} \boldsymbol{\omega}^{\mathbf{m}}$ of $\dfrac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!}\, \dfrac{x^{|\mathbf{m}|}}{|\mathbf{m}|!}$ is integral for every $\mathbf{m}$, because the cohomology classes $\omega_{g,n}(a)$ are integral for $g = 0$.

6. The tensor product for Euler fields and flat identities

In this section, we extend the operation of forming the tensor product to the additional structures of an Euler field and an identity. In order to achieve this, we first rewrite the quasi–homogeneity condition and the defining relation for an identity in terms of operadic correlation functions.

6.1. Quasi–homogeneity condition in terms of correlation functions.

6.1.1. Lemma. *In terms of the abstract correlation functions Y_n the quasi–homogeneity condition (1.11) is given by*

$$\sum_{a\in A}(\sum_{i=1}^{n} d_{a_i a} Y_n(\partial_{a_1}\otimes\cdots\otimes\widehat{\partial_{a_i}}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a)+r^a Y_{n+1}(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a)) \\ =(d_0+D)Y_n(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}). \quad (6.1)$$

Proof. Applying the vector field E in the form (1.8) to (1.6) and making a coefficient check yields (6.1).

6.1.2. Lemma. *The correlation functions (1.17) obey the following relation. For a given n–tree τ:*

$$\sum_{a\in A}(\sum_{f\in F_\tau} d_{fa} Y(\tau)((\bigotimes_{f'\in F_\tau\setminus\{f\}}\partial_{f'})\otimes\partial_a)+r^a Y(\pi^*(\tau))((\bigotimes_{f\in F_\tau}\partial_f)\otimes\partial_a)) \\ =|V_\tau|(D+d_0)Y(\tau)(\bigotimes_{f\in F_\tau}\partial_f). \quad (6.2)$$

Proof. Recall that by definition $Y(\pi^*(\tau))=\sum_{v\in V_\tau} Y(\tau_v^{n+1})$. By applying (6.1) at every vertex v of τ, we obtain

$$\begin{aligned}
&\sum_{f\in F_\tau}\sum_a d_{fa}\, Y(\tau)((\bigotimes_{f'\in F_\tau\setminus\{f\}}\partial_f)\otimes\partial_a) \\
&\quad=\sum_{v\in V_\tau}[\sum_{f\in F_\tau(v)} d_{fa}\bigotimes_{v'\in V_\tau}(Y_{|F_\tau(v')|})((\bigotimes_{f'\in F_\tau\setminus\{f\}}\partial_{f'})\otimes\partial_a)] \\
&\quad=\sum_{v\in V_\tau}[(D+d_0)Y(\tau)(\bigotimes_{f\in F_\tau}\partial_f)-\sum_{a\in A} r^a Y(\tau_v^{n+1})((\bigotimes_{f\in F_\tau}\partial_f)\otimes\partial_a)] \qquad (6.3) \\
&\quad=|V_\tau|(D+d_0)Y(\tau)(\bigotimes_{f\in F_\tau}\partial_f)-\sum_{a\in A} r^a Y(\pi^*(\tau))((\bigotimes_{f\in F_\tau}\partial_f)\otimes\partial_a).
\end{aligned}$$

6.1.3. Proposition. *For the operadic correlation functions $\{Y(\tau)\}$ the quasi–homogeneity condition is equivalent to*

$$\sum_{i=1}^{n}\sum_{a\in A} d_{a_i a} Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\widehat{\partial_{a_i}}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a) - |E_\tau|\, d_0 Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n})$$
$$+\sum_{a\in A} r^a Y(\pi^*(\tau))(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a) = (d_0+D)Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}). \quad (6.4)$$

Proof. Writing out the Casimir elements $\Delta = \sum \partial_p g^{pq}\partial_q$ and applying Lemma 6.1.2 yields:

$$\sum_{i=1}^{n}\sum_{a\in A} d_{a_i a} Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\widehat{\partial_{a_i}}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a) + |E_\tau|\, D\, Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n})$$
$$= \sum_{i=1}^{n}\sum_{a\in A} d_{a_i a} Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\widehat{\partial_{a_i}}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a)$$
$$+ |E_\tau|\, D \sum_{\substack{(p_1,\dots p_{|E_\tau|}) p_i\in A\\ (q_1,\dots q_{|E_\tau|}) q_i\in A}} (\bigotimes_{v\in V_\tau} Y_{F_v})(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}\otimes\bigotimes_{j=1}^{|E_\tau|}(\partial_{p_j} g^{p_j q_j}\otimes\partial_{q_i}))$$
$$\overset{(*)}{=} \sum_{\substack{(p_1,\dots p_{|E_\tau|}) p_i\in A\\ (q_1,\dots q_{|E_\tau|}) q_i\in A}} [\sum_{i=1}^{n}\sum_{a\in A} d_{a_i a} Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\widehat{\partial_{a_i}}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a\otimes\bigotimes_{j=1}^{|E_\tau|}(\partial_{p_j} g^{p_j q_j}\otimes\partial_{q_i}))$$
$$+ \sum_{i=1}^{|E_\tau|}\sum_{a\in A} d_{p_i a}(\bigotimes_{v\in V_\tau} Y_{F_v})(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}\otimes\bigotimes_{j=1, j\neq i}^{|E_\tau|}(\partial_{p_j} g^{p_j q_j}\otimes\partial_{q_i})\otimes\widehat{\partial_{p_i}} g^{p_i a}\partial_{q_i}\otimes\partial_a)$$
$$+ \sum_{i=1}^{|E_\tau|}\sum_{a\in A} d_{p_j a}(\bigotimes_{v\in V_\tau} Y_{F_v})(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}\otimes\bigotimes_{i=1, i\neq j}^{|E_\tau|}(\partial_{p_i} g^{p_i q_i}\otimes\partial_{q_i})\otimes\partial_{p_i} g^{p_i a}\widehat{\partial_{q_i}}\otimes\partial_a)]$$
$$= (|E|_\tau + 1)(D+d_0)Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n})$$
$$- \sum_{a\in A} r^a Y(\pi^*(\tau))(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a). \quad (6.5)$$

The equality $(*)$ holds due to (1.9). Rewriting (6.5), we obtain (6.4). Vice versa postulating (6.4), we see that it reduces to (6.1) for the one–vertex tree (ρ_n).

6.2. The identity in terms of correlation functions. As previously remarked, we will assume that the identity is a flat vector field $e = \partial_0$. As the semi–simplicity of E this restriction is satisfied in the case of quantum cohomology.

6.2.1. Remark. From Corollary 2.1.1 of [M2], we have that

$$Y_3(\partial_a, \partial_b, \partial_0) = g_{ab} \tag{6.6}$$

and

$$Y_n(\partial_{a_1} \otimes \cdots \otimes \partial_{a_{n-1}} \otimes \partial_0) = 0 \qquad \forall n > 3 \tag{6.7}$$

are equivalent to the fact that ∂_0 is a flat identity.

In terms of operadic ACFs one obtains:

6.2.2. Proposition. *For a flat identity $e = \partial_0$ and for any stable n–tree τ with $n > 3$*

$$Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_{n-1}} \otimes \partial_0) = Y(\pi_*(\tau))(\partial_{a_1} \otimes \cdots \otimes \partial_{a_{n-1}}). \tag{6.8}$$

Proof. From (6.7) we know that $Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_{n-1}} \otimes \partial_0) = 0$, if the valence of the vertex v_0 with the tail marked with n is greater than three or, in other words, if the vertex remains stable after forgetting the tail n. Assume now that the vertex has valence three. Noticing that for a flat identity $Y_3(\partial_a, \partial_b, \partial_0) = g_{ab}$ the result follows by direct calculation. There are two cases: either v_0 has two tails marked n and i for some i and is joined to one other vertex v' by the edge e or v_0 just has one tail and is joined to two other vertices by the edges e_1 and e_2. In the first case we get

$$\begin{aligned}
&Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_{n-1}} \otimes \partial_0 \otimes \Delta^{\otimes |E_\tau|}) \\
&\quad = \Big(\bigotimes_{v \in V_\tau \setminus \{v_0\}} \Big(\bigotimes_{f \in F_\tau(v)} Y_{F_\tau(v)}\Big) \otimes Y_{F_\tau(v_0)}\Big) \\
&\qquad (\partial_{a_1} \otimes \cdots \otimes \widehat{\partial_{a_i}} \otimes \cdots \otimes \partial_{a_{n-1}} \otimes \Delta^{\otimes |E_\tau| - 1} \otimes \Delta_e \otimes \partial_{a_i} \otimes \partial_0) \\
&\quad = \sum_{pq} \Big(\bigotimes_{v \in V_\tau \setminus \{v_0\}} \Big(\bigotimes_{f \in F_\tau(v)} Y_{F_\tau(v)}\Big)\Big) \\
&\qquad (\partial_{a_1} \otimes \cdots \otimes \widehat{\partial_{a_i}} \otimes \cdots \otimes \partial_{a_{n-1}} \otimes \Delta^{\otimes |E_\tau| - 1} \otimes \partial_p g^{pq} g_{q a_i}) \\
&\quad = \Big(\bigotimes_{v \in V_\tau \setminus \{v_0\}} \Big(\bigotimes_{f \in F_\tau(v)} Y_{F_\tau(v)}\Big)\Big)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_{n-1}} \otimes \Delta^{\otimes |E_\tau| - 1}) \\
&\quad = Y(\pi_*(\tau))(\partial_{a_1} \otimes \cdots \otimes \partial_{a_{n-1}})
\end{aligned}$$

likewise in the second case

$$
\begin{aligned}
&Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\partial_0\otimes\Delta^{\otimes|E_\tau|})\\
&\quad=(\bigotimes_{v\in V_\tau\setminus\{v_0\}}(\bigotimes_{f\in F_\tau(v)}Y_{F_\tau(v)})\otimes Y_{F_\tau(v_0)})\\
&\quad\quad(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\Delta^{\otimes|E_\tau|-2}\otimes\Delta_{e_1}\otimes\Delta_{e_2}\otimes\partial_0)\\
&\quad=\sum_{pq,rs}(\bigotimes_{v\in V_\tau\setminus\{v_0\}}(\bigotimes_{f\in F_\tau(v)}Y_{F_\tau(v)}))\\
&\quad\quad(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\Delta^{\otimes|E_\tau|-2}\otimes\partial_p g^{pq}g_{qr}g^{rs}\otimes\partial_s)\\
&\quad=(\bigotimes_{v\in V_\tau\setminus\{v_0\}}(\bigotimes_{f\in F_\tau(v)}Y_{F_\tau(v)}))(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\Delta^{\otimes|E_\tau|-1})\\
&\quad=Y(\pi_*(\tau))(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}).
\end{aligned}
$$

6.2.3. Remark. In the setting of operads and higher order multiplications ([Ge1],[GK]), the formulas (6.6) and (6.7) for a flat identity $e=\partial_0$ correspond to the statements that e is an identity for $\circ_2$ and acts as a zero for all higher multiplications $\circ_n, n\geq 3$. The contents of Proposition 6.2.2 is the extension of these properties to any concatenation of these multiplications.

After these preparations, we come to the main result of this section:

6.3. Theorem. *Given two formal Frobenius manifolds (H',g',Φ') and (H'',g'',Φ'') with Euler fields*

$$E'=\sum_{a'b'\in A'}d'_{a'b'}x'^{a'}\partial'_{b'}+\sum_{a'\in A'}r'^{a'}\partial'_{a'}\qquad\text{of weight } D' \text{ and}\tag{6.9}$$

$$E''=\sum_{a''b''\in A''}d''_{a''b''}x''^{a''}\partial''_{b''}+\sum_{a''\in A''}r''^{a''}\partial''_{a''}\qquad\text{of weight } D''\tag{6.10}$$

and with flat identities e',e'' of the same weight $d'_0=d''_0=d$, then

$$e=\partial'_{00}=e'\otimes e''\tag{6.11}$$

and

$$E=\sum_{(b',b'')\in A'\times A''}[\sum_{a'\in A'}(d'_{a'b'}x^{a'b''})+\sum_{a''\in A''}(d''_{a''b''}x^{b'a''})-dx^{b'b''}]\,\partial_{b'b''}+\sum_{a'\in A'}r'^{a'}\partial_{a'0}+\sum_{a''\in A''}r''^{a''}\partial_{0a''}\tag{6.12}$$

define a flat identity of weight d and an Euler field of weight $D'+D''-2d$ on the tensor product (H,g,Φ) of (H',g',Φ') and (H'',g'',Φ'').

Before we can prove the above theorem, we need one more Lemma about the properties of the diagonal class $\Delta_{\overline{M}_{0n}}$.

6.4. Lemma.
$$(id, \pi_*)(\Delta_{\overline{M}_{0n}}) = (\pi^*, id)(\Delta_{\overline{M}_{0n-1}}) \tag{6.13}$$
and
$$(\pi_*, \pi_*)(\Delta_{\overline{M}_{0n}}) = 0. \tag{6.14}$$

Proof. Consider two any strata classes $D_\tau \in A^*(\overline{M}_{0n}), D_\sigma \in A^*(\overline{M}_{0n-1})$. Using the projection formula twice, we obtain
$$\int_{\overline{M}_{0n}} D_\tau \cup \pi^*(D_\sigma) = \int_{\overline{M}_{0n-1}} \pi_*(D_\tau) \cup D_\sigma$$
$$\Leftrightarrow \int_{\overline{M}_{0n}\times\overline{M}_{0n}} (D_\tau \boxtimes \pi^*(D_\sigma)) \cup \Delta_{\overline{M}_{0n}} = \int_{\overline{M}_{0n-1}\times\overline{M}_{0n-1}} (\pi_*(D_\tau) \boxtimes D_\sigma) \cup \Delta_{\overline{M}_{0n-1}}$$
$$\Leftrightarrow \int_{\overline{M}_{0n}\times\overline{M}_{0n-1}} (D_\tau \boxtimes D_\sigma) \cup (id, \pi_*)\Delta_{\overline{M}_{0n}} = \int_{\overline{M}_{0n}\times\overline{M}_{0n-1}} (D_\tau \boxtimes D_\sigma) \cup (\pi^*, id)\Delta_{\overline{M}_{0n-1}}.$$
Since the intersection pairing is non–degenerate and the classes $D_\tau \boxtimes D_\sigma$ generate $A^*(\overline{M}_{0n-1} \times \overline{M}_{0n})$, the formula (6.13) follows. Using the same type of argument for
$$\int_{\overline{M}_{0n-1}\times\overline{M}_{0n-1}} (D_\tau \boxtimes D_\sigma) \cup (\pi_*, \pi_*)\Delta_{\overline{M}_{0n}}$$
$$= \int_{\overline{M}_{0n-1}\times\overline{M}_{0n-1}} (\pi^*, \pi^*)(D_\tau \boxtimes D_\sigma) \cup \Delta_{\overline{M}_{0n}}$$
$$= \int_{\overline{M}_{0n}} \pi^*(D_\tau) \cup \pi^*(D_\sigma) = \int_{\overline{M}_{0n}} \pi^*(D_\tau \cup D_\sigma) = 0$$
where the last zero is due to dimensional reasons, we obtain the second claim (6.14).

Proof of the Theorem.

As in 4.3.2, we choose the coordinates $x^{a'a''}$ corresponding to the basis $\partial_{a'} \otimes \partial_{a''}$. The metric for the tensor product is given by
$$g_{a'b',a''b''} := g(\partial_{a'} \otimes \partial_{a''}, \partial_{b'} \otimes \partial_{b''}) = g'(\partial_{a'}, \partial_{b'})g''(\partial_{a''}, \partial_{b''}) = g'_{a',b'}g''_{a'',b''}. \tag{6.15}$$

Euler field.

First we check that E is conformal of weight $D' + D'' - 2d$. On the basis of flat vector fields we calculate:

$$\begin{aligned} &g([\partial_{a'a''}, E], \partial_{b'b''}) + g(\partial_{a'a''}, [\partial_{b'b''}, E]) \\ &= \sum_{c'} d'_{a'c'} g'_{c'b'} g''_{a''b''} + \sum_{c''} d''_{a''c''} g'_{c'b'} g''_{c''b''} + \sum_{c'} d'_{b'c'} g'_{a'c'} g''_{a''b''} + \sum_{c''} d''_{b''c''} g'_{a'b'} g''_{a''c''} \\ &\quad - 2dg'a'b'g''_{a''b''} \\ &= (D' + D'' - 2d) g_{a'a'',b'b''}. \end{aligned} \tag{6.16}$$

We will prove the fact that E is indeed an Euler field by verifying the quasi–homogeneity condition (1.11).

Set $D = D' + D'' - 2d$ and $\gamma = \sum x^{a'a''}\partial'_{a'} \otimes \partial''_{a''}$:

$$
\begin{aligned}
E_1\Phi(\gamma) &= E_1 \sum_{n\geq 3} \frac{1}{n!} Y_n(\gamma^{\otimes n}) = E_1 \sum_{n\geq 3} \frac{1}{n!} (Y' \otimes Y'')(\Delta_{\overline{M}_{0n}})(\gamma^{\otimes n}) \\
&= \sum_{n\geq 3} \frac{1}{n!} x^{a'_n a''_n} \cdots x^{a'_1 a''_1} \Big(\sum_{a' \in A'} \sum_{i=1}^{n} d'_{a'_i a'} (Y' \otimes Y'')(\Delta_{\overline{M}_{0n}}) \\
&\qquad ((\partial'_{a'_1} \otimes \partial''_{a''_1}) \otimes \cdots \otimes (\widehat{\partial'_{a'_i} \otimes \partial''_{a''_i}}) \otimes \cdots \otimes (\partial'_{a'_n} \otimes \partial''_{a''_n}) \otimes (\partial'_{a'} \otimes \partial''_{a''_i})) \\
&\quad + \sum_{a'' \in A''} \sum_{i=1}^{n} d''_{a''_i a''} (Y' \otimes Y'')(\Delta_{\overline{M}_{0n}}) \\
&\qquad ((\partial'_{a'_1} \otimes \partial''_{a''_1}) \otimes \cdots \otimes (\widehat{\partial'_{a'_i} \otimes \partial''_{a''_i}}) \otimes \cdots \otimes (\partial'_{a'_n} \otimes \partial''_{a''_n}) \otimes (\partial'_{a'_i} \otimes \partial''_{a''})) \\
&\quad - n\, d\, (Y' \otimes Y'')(\Delta_{\overline{M}_{0n}})((\partial'_{a'_1} \otimes \partial''_{a''_1}) \otimes \cdots \otimes (\partial'_{a'_n} \otimes \partial''_{a''_n}))\Big) \\
&\overset{(*)}{=} \sum_{n\geq 3} \frac{1}{n!} x^{a'_n a''_n} \cdots x^{a'_1 a''_1} \Big((D' + D'' - d)(Y' \otimes Y'')(\Delta_{\overline{M}_{0n}}) \\
&\qquad ((\partial'_{a'_1} \otimes \partial''_{a''_1}) \otimes \cdots \otimes (\partial'_{a'_n} \otimes \partial''_{a''_n})) \\
&\quad - \sum_{a' \in A'} r'^{a'} (Y' \otimes Y'')((\pi^*, id)(\Delta_{\overline{M}_{0n}}))((\partial'_{a'_1} \otimes \partial''_{a''_1}) \otimes \cdots \otimes (\partial'_{a'_n} \otimes \partial''_{a''_n}) \otimes \partial'_{a'}) \\
&\quad - \sum_{a'' \in A''} r''^{a''} (Y' \otimes Y'')((id, \pi^*)(\Delta_{\overline{M}_{0n}}))((\partial'_{a'_1} \otimes \partial''_{a''_1}) \otimes \cdots \otimes (\partial'_{a'_n} \otimes \partial''_{a''_n}) \otimes \partial''_{a''})\Big) \\
&= \sum_{n\geq 3} \frac{1}{n!} \Big((D + d)(Y' \otimes Y'')(\Delta_{\overline{M}_{0n}})(\gamma^{\otimes n}) \\
&\quad - \sum_{a' \in A'} r'^{a'} (Y' \otimes Y'')((\pi^*, id)(\Delta_{\overline{M}_{0n}}))(\gamma^{\otimes n} \otimes \partial'_{a'}) \\
&\quad - \sum_{a'' \in A''} r''^{a''} (Y' \otimes Y'')((id, \pi^*)(\Delta_{\overline{M}_{0n}}))(\gamma^{\otimes n} \otimes \partial''_{a''})\Big).
\end{aligned}
\tag{6.17}
$$

To obtain $(*)$ write $\Delta_{\overline{M}_{0n}} = \sum \boldsymbol{\tau} g^{\boldsymbol{\tau}\boldsymbol{\sigma}} \otimes \boldsymbol{\sigma}$ as in 4.1 and apply Proposition 6.1.3 to both tensor factors of each summand. Furthermore, notice that the $\boldsymbol{\tau}, \boldsymbol{\sigma}$ are homogeneous and $g^{\boldsymbol{\sigma}\boldsymbol{\tau}} = 0$ unless $\boldsymbol{\sigma} \otimes \boldsymbol{\tau} \in V(\Gamma_{n,e}) \otimes V(\Gamma_{n,n-3-e})$ (4.2).

On the other hand, applying Proposition 6.2.2, we obtain up to quadratic terms

$$
\begin{aligned}
E^0\Phi(\gamma) = &\sum_{n\geq 3}\frac{1}{(n-1)!}\Big(\sum_{a'\in A'} r'^{a'}(Y'\otimes Y'')(\Delta_{\overline{M}_{0n}})(\gamma^{\otimes n-1}\otimes\partial'_{a'}\otimes\partial''_0)\\
&+\sum_{a''\in A''} r''^{a''}(Y'\otimes Y'')(\Delta_{\overline{M}_{0n}})(\gamma^{\otimes n-1}\otimes\partial'_0\otimes\partial''_{a''})\Big)\\
= &\sum_{n\geq 3}\frac{1}{n!}\Big(\sum_{a'\in A'} r'^{a'}(Y'\otimes Y'')(id,\pi_*)(\Delta_{\overline{M}_{n+1}})(\gamma^{\otimes n}\otimes\partial'_{a'})\\
&+\sum_{a''\in A''}(Y'\otimes Y'')(\pi_*,id)(\Delta_{\overline{M}_{n+1}})(\gamma^{\otimes n}\otimes\partial''_{a''})\Big) \qquad (6.18)
\end{aligned}
$$

Applying the formula (6.13), we see that the sum of (6.17) and (6.18) is just the the quasi–homogeneity condition for E and therefore E is an Euler field.

Identity.

The proposed identity $\partial'_0\otimes\partial''_0$ is a flat field by definition. Furthermore,

$$
\begin{aligned}
Y_3(\partial'_{a'}\otimes\partial''_{a''}\otimes\partial'_{b'}\otimes\partial''_{b''}\otimes\partial'_0\otimes\partial''_0) &= Y'_3(\partial'_{a'}\otimes\partial'_{b'}\otimes\partial'_0)Y''_3(\partial''_{a''}\otimes\partial''_{b''}\otimes\partial''_0)\\
&= g_{a'b',a''b''} \qquad (6.19)
\end{aligned}
$$

and for $n\geq 3$ by Proposition 6.2.2 and (6.14)

$$
\begin{aligned}
&Y_n((\partial'_{a'_1}\otimes\partial''_{a''_1})\otimes\cdots\otimes(\partial'_{a'_{n-1}}\otimes\partial''_{a''_{n-1}})\otimes(\partial'_0\otimes\partial''_0))\\
&\quad=(Y'\otimes Y'')((\pi_*,\pi_*)\Delta_{\overline{M}_{0n}})((\partial'_{a'_1}\otimes\partial''_{a''_1})\otimes\cdots\otimes(\partial'_{a'_{n-1}}\otimes\partial''_{a''_{n-1}}))\\
&\quad=0 \qquad (6.20)
\end{aligned}
$$

which proves that $\partial'_0\otimes\partial''_0$ is indeed an identity by Remark 6.2.1. The weight of this identity can be read off the Euler field as $d+d-d=d$, proving the theorem.

6.5. Remarks. The condition that the weights of the identities are equal can be met by a rescaling of the Euler fields as long as not only one of the weights is 0. In the following, we will always assume this when considering the tensor product.

Since, given a metric and the multiplication on the fibers of a Frobenius manifold, the identity is uniquely determined —cf .[M2]—, the above identity is the only identity compatible with the choice of the tensor metric (6.15).

The theorem, however, contains no such uniqueness property for the Euler field, but there are several reasons for the choice of this particular type of Euler field. If the E_1–part is regarded as providing the operator $\mathcal{V}$ of (1.10), then our choice of E_1 for the tensor product is equivalent up to the shift by d which is necessary to accommodate the dependence of the tensor product on the diagonal in $H^*(\overline{M}_{0n}\times\overline{M}_{0n})$ to the natural definition:

$$\mathcal{V} := \mathcal{V}'\otimes\mathrm{id}+\mathrm{id}\otimes\mathcal{V}''. \qquad (6.21)$$

As remarked in [M2], if the action of $\mathrm{ad}(E)$ is semi–simple on H, there is a natural grading of H induced by the action of $\mathrm{ad}(E)$, shifted by d_0. This grading basically fixes the E_1 component. In the setting of quantum cohomology, this grading is just (half) the usual grading for the cohomology groups. The additivity is just the fact that under the Künneth formula the total degree of a class is the sum of the degrees of the two components. The natural grading on the space of $H' \otimes H''$ is consequently given by the grading operator $\mathrm{ad}(E' \otimes \mathrm{id} + \mathrm{id} \otimes E'')$ shifted by d, so that the tensor product of ∂'_a and ∂''_b of degrees $\delta'_{a'}$ and $\delta''_{a''}$ is of degree $\delta'_{a'} + d + \delta''_{a''} + d - d$. Recalling that d_a was the eigenvalue of $-\mathrm{ad}(E)$, we obtain $d_{a'a''} = d'_{a'} + d''_{a''} - d$.

In the physical realm of topological field theories [DVV], the above argument for the choice of E_1 just reflects the additivity of a $U(1)$ charge.

The choice for E_0 is motivated by quantum cohomology where the E_0–part corresponds to the canonical class. Thus, the definition of $E_0 = E'_0 \otimes \partial''_0 + \partial'_0 \otimes E''_0$ corresponds to the formula $K_{X\times Y} = K_X \otimes 1 + 1 \otimes K_Y$. More generally, it corresponds to the map $H^*(V) \times H^*(W) \to H^*(V \times W) : (v, w) \to pr_1^*(v) + pr_2^*(w)$ which can be extended to the Frobenius structure, cf. Section 7.2.

Furthermore, in view of (6.17) and Lemma 6.4, E_0 seems to be the only possible choice, if one postulates (6.21).

7. Germs of pointed Frobenius manifolds and the tensor product

First, we consider the definition of the tensor product of two germs of pointed Frobenius manifolds by the passage via formal Frobenius manifolds and the possible obstructions. Then assuming convergence, we investigate the dependence of this procedure on the chosen base–points.

7.1. The tensor product. Given two germs of pointed Frobenius manifolds, we can define their tensor product in the category of formal Frobenius manifolds. To retrieve an honest germ again, however, we additionally have to postulate the convergence of the tensor potential.

7.1.1. Definition. Given two germs of pointed Frobenius manifolds $(\mathbf{M}', m_0')$ and $(\mathbf{M}'', m_0'')$, let (H', g', Φ') and (H'', g'', Φ'') be the associated formal Frobenius manifolds. If the tensor product (H, g, Φ) of (H', g', Φ') and (H'', g'', Φ'') has a convergent potential, we define the tensor product $(\mathbf{M}, m_0)$ of $(\mathbf{M}', m_0')$ and $(\mathbf{M}'', m_0'')$ as the associated germ of a pointed Frobenius manifold.

7.1.2. Remark. As will be discussed in the next paragraph, the convergence condition is automatic in the case of semi–simple Frobenius manifolds, if the base–point is tame, i.e. the spectrum of the multiplication with E is non–degenerate at this point.

7.2. Base–point dependence of the tensor product. The above definition of tensor product depends on the choice of the base–point, since this choice determines the ACFs which will be tensored and, as we already noted in Section 2.3, the choice of a different zero in flat coordinates will lead to a different system of ACFs (see (2.3)) and thus to a different germ.

7.2.1. Lemma. *Let $\{Y_n\}$ be the ACFs corresponding to the base–point m_0 as zero in x–coordinates and $\{\widehat{Y}_n\}$ be the ACFs corresponding to a new base–point $\widehat{m}_0$ which lies inside the domain of convergence of the potential with x–coordinates $x^a(\widehat{m}_0) = x_0^a$, see (2.3), then the operadic correlation functions transform in the following way:*

For any stable n–tree τ:

$$\widehat{Y}(\tau)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) = \sum_{N \geq 0} \frac{1}{N!} \sum_{(b_1, \ldots, b_N): b_i \in A} \epsilon(b|a)\, x_0^{b_N} \cdots x_0^{b_1}$$
$$Y(\pi^*_{\{n+1, \ldots, n+N\}}(\tau))(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \partial_{b_1} \otimes \cdots \otimes \partial_{b_N}). \quad (7.1)$$

Proof. Inserting (2.3) into the definition of $Y(\tau)$ (1.15), we see that the correlation functions having a prefactor $x_0^{b_N} \cdots x_0^{b_1}$ are those belonging to trees with $N - n$ tails added in an arbitrary fashion to τ. The sum over all of these trees is just $\pi^*_{\{n+1, \ldots, n+N\}}(\tau)$, whence the Lemma follows.

In order to investigate the base–point dependence of the tensor product, we need a Lemma about the diagonal $\Delta_{\overline{M}_{0S}}$ which extends Lemma 6.4.

7.2.2. Lemma. *For any two disjoint subsets $S, T \subset \{1, \dots, n\}$*

$$(\pi_S^*, \pi_T^*)(\Delta_{\overline{M}_{0\{1,\dots,n\}\setminus(S\cup T)}}) = (\pi_{T*}, \pi_{S*})(\Delta_{\overline{M}_{0n}}). \tag{7.2}$$

Proof. Writing (π_S^*, π_T^*) as $(\pi_S^*, id) \circ (id, \pi_T^*)$, we obtain, after repeated application of Lemma 6.4 in an appropriate version, that

$$(\pi_S^*, \pi_T^*)(\Delta_{\overline{M}_{0\{1,\dots,n\}\setminus(S\cup T)}}) = (\pi_S^* \circ \pi_{T*}, id)(\Delta_{\overline{M}_{0\{1,\dots,n\}\setminus S}}).$$

Since π_S and π_T commute if $T \cap S = \emptyset$, we can prove the equality (7.2) again by Lemma 6.4.

This Lemma enables us to prove that tensoring at the points m_0' and m_0'' and then shifting the base–point to $\widehat{m}_0$ with non–zero x–coordinates $x^{0a''}(\widehat{m}_0) = x_0''^{a''}, x^{a'0}(\widehat{m}_0) = x_0'^{a'}$ corresponds to tensoring at the shifted base–points $\widehat{m}_0'$ with m_0'–coordinates $x_0'^{a'}$ and $\widehat{m}_0''$ with m_0''–coordinates $x_0''^{a''}$.

More precisely, let $(\mathbf{M}', m_0')$ and $(\mathbf{M}'', m_0'')$ be two germs of pointed Frobenius manifolds. Assume that their tensor product as germs of pointed Frobenius manifolds according to 7.1.1 exists and denote it by $(\mathbf{M}, m_0)$. Additionally, consider the two germs of pointed Frobenius manifolds $(\mathbf{M}', \widehat{m}_0')$ and $(\mathbf{M}'', \widehat{m}_0'')$ obtained from $(\mathbf{M}', m_0')$ and $(\mathbf{M}'', m_0'')$ by shifting the base–points to $\widehat{m}_0'$ and $\widehat{m}_0''$ inside the domain of convergence of Φ' and Φ'' respectively and assume that their tensor product $(\widetilde{\mathbf{M}}, \widetilde{m}_0)$ exists. Let the shifted base–point $\widehat{m}_0'$ have the coordinates $x_0'^{a'}$ in $(\mathbf{M}', m_0')$ and $\widehat{m}_0''$ have coordinates $x_0''^{a''}$ in $(\mathbf{M}'', m_0'')$. Finally, let $(\mathbf{M}, \widehat{m}_0)$ be the germ obtained from $(\mathbf{M}, m_0)$ by shifting the base–point to the point $\widehat{m}_0$ whose non–zero coordinates are $x^{0a''}(\widehat{m}_0) = x_0''^{a''}$ and $x^{a'0}(\widehat{m}_0) = x_0'^{a'}$ for $a', a'' \neq 0$ and $x^{00}(\widehat{m}_0) = x_0'^0 + x_0''^0$, if this point is inside the domain of convergence of Φ.

7.2.3. Theorem. *With the notations and conditions as stated above, the two germs of pointed Frobenius manifolds $(\mathbf{M}, \widehat{m}_0)$ and $(\widetilde{\mathbf{M}}, \widetilde{m}_0)$ are isomorphic. Furthermore, if $(\mathbf{M}', m_0')$ and $(\mathbf{M}'', m_0'')$ also carry Euler fields and flat identities, then the corresponding structures $\widehat{E}, \widehat{e}$ on $(\mathbf{M}, \widehat{m}_0)$ and $\widetilde{E}, \widetilde{e}$ on $(\widetilde{\mathbf{M}}, \widetilde{m}_0)$ are identified under the isomorphism.*

Proof. Denote the ACFs corresponding to $(\mathbf{M}', m_0')$ by $\{Y_n'\}$ and the appropriate coordinates by x'. Likewise denote the ACFs corresponding to $(\mathbf{M}'', m_0'')$ by $\{Y_n''\}$ and the appropriate coordinates by x''. Using the Definition 7.1.1 and then shifting to the new base–point with non–zero coordinates $x^{0a''}(\widehat{m}_0) = x_0''^{a''}$ and $x^{a'0}(\widehat{m}_0) = x_0'^{a'}$ in the coordinates x of the tensor product denoting the correlation functions of the tensor product by $\{Y_n\}$, we obtain the following formula for the shifted ACFs $\{\widehat{Y}_n\}$ given by the new base–point:

$$
\begin{aligned}
&\widehat{Y}_n(\partial_{a'_1a''_1}\otimes\cdots\otimes\partial_{a'_na''_n})\\
&=\sum_{N\geq 0}\frac{1}{N!}\sum_{l=0}^{N}\sum_{\substack{(b'_1,\dots,b'_l)|b'\in A'\\(b''_1,\dots,b''_{N-l})|b''\in A''}}\binom{N}{l}\epsilon(b'0|a'a'')\epsilon(0b''|a'a'')\,x_0^{\prime\prime b''_{N-l}}\cdots x_0^{\prime\prime b''_1}x_0^{\prime b'_l}\cdots x_0^{\prime b'_1}\\
&(Y'\otimes Y'')(\Delta_{\overline{M}_{0,N+n}})(\partial_{a'_1a''_1}\otimes\cdots\otimes\partial_{a'_na''_n}\otimes\partial_{b'_10}\otimes\dots\partial_{b'_l0}\otimes\partial_{0b''_1}\otimes\cdots\otimes\partial_{0b''_{N-l}})\\
&\qquad=\sum_{N\geq 0}\frac{1}{N!}\sum_{l=0}^{N}\sum_{\substack{(b'_1,\dots,b'_l)|b'\in A'\\(b''_1,\dots,b''_{N-l})|b''\in A''}}\binom{N}{l}\epsilon(b'|a')\epsilon(b''|a'')\epsilon(b'|a'')\epsilon(b''|a')\\
&\quad x_0^{\prime\prime b''_{N-l}}\cdots x_0^{\prime\prime b''_1}x_0^{\prime b'_l}\cdots x_0^{\prime b'_1}(Y'\otimes Y'')((\pi_{\{n+l+1,\dots,n+N\}*},\pi_{\{n+1,\dots,n+l\}*})(\Delta_{\overline{M}_{0,N+n}}))\\
&\qquad\qquad(\partial_{a'_1a''_1}\otimes\cdots\otimes\partial_{a'_na''_n}\otimes\partial_{b'_1}\otimes\dots\partial_{b'_l}\otimes\partial_{b''_1}\otimes\cdots\otimes\partial_{b''_{N-l}})
\end{aligned}
\tag{7.3}
$$

due to Proposition 6.2.2. On the other hand, tensoring the ACFs $\{\widehat{Y}'_n\}$ and $\{\widehat{Y}''_n\}$ which corresponds to shifting to the points $\widehat{m}'_0$ and $\widehat{m}''_0$ and then performing the tensor product, and then utilizing Lemma 7.2.1 yields the result:

$$
\begin{aligned}
&\widetilde{Y}_n(\partial_{a'_1a''_1}\otimes\cdots\otimes\partial_{a'_na''_n})=(\widehat{Y}'\otimes\widehat{Y}'')(\Delta_{\overline{M}_{0n}})(\partial_{a'_1a''_1}\otimes\cdots\otimes\partial_{a'_na''_n})\\
&\qquad=\sum_{N\geq 0}\sum_{l=0}^{N}\sum_{\substack{(b'_1,\dots,b'_l)|b'\in A'\\(b''_1,\dots,b''_{N-l})|b''\in A''}}\frac{1}{N!(N-l)!}\epsilon(b'|a')\epsilon(b''|a'')\epsilon(b'|a'')\epsilon(b''|a')\\
&\quad\times x_0^{\prime\prime b''_{N-l}}\cdots x_0^{\prime\prime b''_1}x_0^{\prime b'_l}\cdots x_0^{\prime b'_1}(Y'\otimes Y'')((\pi^*_{\{n+1,\dots,n+l\}},\pi^*_{\{n+l+1,\dots,n+N\}})(\Delta_{\overline{M}_{0,N+n}}))\\
&\qquad\qquad(\partial_{a'_1a''_1}\otimes\cdots\otimes\partial_{a'_na''_n}\otimes\partial_{b'_1}\otimes\dots\partial_{b'_l}\otimes\partial_{b''_1}\otimes\cdots\otimes\partial_{b''_{N-l}}).
\end{aligned}
\tag{7.4}
$$

Applying Lemma 7.2.2 with $S=\{n=1,\dots,n+l\}, T=\{n+l+1,\dots,n+N\}$, we see that (7.3) and (7.4) and thus the multiplications, respectively the potentials up to quadratic terms, coincide. Since the flat structures and the metrics are not altered by a shift in the base–point, the first part of the theorem follows.

The equation for the identities is clear, since flat vector fields remain invariant under a different choice of zero.

For the Euler fields, we find, keeping the notation of Theorem 6.3 and identifying the coordinates of $(\mathbf{M},\widehat{m}_0)$ and $(\widetilde{\mathbf{M}},\widetilde{m}_0)$:

$$
\begin{aligned}
\widehat{E} = & \sum_{(b',b'')\in A'\times A''} [\sum_{a'\in A'} (d'_{a'b'} x^{a'b''}) + \sum_{a''\in A''} (d''_{a''b''} x^{b'a''}) - dx^{b'b''}]\, \partial_{b'b''} \\
& + \sum_{a'\in A'} r'^{a'} \partial_{a'0} + \sum_{a''\in A''} r''^{a''} \partial_{0a''} \\
& - \sum_{a'\in A'} dx_0'^{a} \partial_{a'0} - \sum_{a''\in A''} dx_0''^{a} \partial_{0a''} \\
& + \sum_{\substack{a'\in A' \\ b'\in A'}} d'_{a',b'} x_0'^{a} \partial_{b'0} + \sum_{\substack{a''\in A'' \\ b''\in A''}} d''_{a'',b''} x_0''^{a} \partial_{0b''} \\
& = \widetilde{E}. \quad \square
\end{aligned}
\tag{7.5}
$$

7.2.4. Remark. Looking at the Theorem 7.2.3 from the viewpoint of deformations of the algebra structure over the base–point, it identifies the possible deformation parameters from the two factors of two tensored multiplications as special directions in the tensor product. We point out that in the presence of a flat identity, the number of possible deformation parameters for a Frobenius manifold $\mathbf{M}$ is $\dim(M) - 1$ where possible deformation parameters are those for which it is not a priori clear that all third derivatives of the potential including a derivative with respect to this parameter are constant. The theorem hence identifies $n+m-2$ of the $nm-1$ possible deformation parameters of the tensor product of $\mathbf{M}^n$ and $\mathbf{M}'^m$ as deformations coming from deforming the algebras before tensoring. In the direction of the identity the multiplication itself is not deformed. The coordinate $x_0^{00} = x_0'^0 + x_0''^0$ is determined by the Euler fields which do depend on the direction corresponding to the identity. The rest of the parameters are new parameters and correspond to "coupling" the algebras.

If one does not deform into these new directions, the resulting multiplication has the simple structure

$$
\partial_{a'} \otimes \partial_{b'} \circ_{\gamma'\otimes e''+e'\otimes\gamma''} \partial_{a''} \otimes \partial_{b''} = (\partial_{a'} \circ_{\gamma'} \partial_{b'}) \otimes (\partial_{a''} \circ_{\gamma''} \partial_{b''}). \tag{7.6}
$$

7.2.5. Corollary. In the setting of Theorem 7.2.3, let γ' and γ'' be points inside the domain of convergence of the potentials of the factors and $\gamma = \gamma' \otimes e'' + e' \otimes \gamma''$ be a point inside the domain of convergence of the tensor potential, then the multiplication over the point γ is simply the tensor product of the multiplications over the points γ' and γ''.

8. The tensor product for semi–simple Frobenius manifolds

8.1. The tensor product for semi–simple Frobenius manifolds with Euler field and flat identity. In Section 7, the operation of forming the tensor product was introduced for germs of pointed Frobenius manifolds under the condition of the convergence of the tensor potential. More precisely, it is stated that in the case of semi–simple Frobenius manifolds with an Euler field and a flat identity, the split cover of the subspace of tame points $\mathbf{M}$ is already determined by any germ near any given tame point $m_0 \in M$ in terms of initial conditions for Schlesinger's equations. In particular, given such a tame point we may extend the potential to a neighborhood of this point. Thus, we may define the tensor product for two germs of tame semi–simple Frobenius manifolds $\mathbf{M}', \mathbf{M}''$ depending on the choice of two tame base–points m_0', m_0'', if the tensor product provides special initial conditions. Since the algebra in the tangent space over the base–point of the tensor product is just the tensor product of two semi–simple algebras, it is itself semi–simple, so the only possible obstruction to obtaining special initial conditions is that the new base–point may not be tame. This condition, however, is not very restrictive, as we will show later. But first, we conditionally define:

8.1.1. Definition. The tensor product of two germs of semi–simple Frobenius manifolds $\mathbf{M}', \mathbf{M}''$ relative to the tame points m_0', m_0'' is defined to be the semi–simple Frobenius manifold given by the initial conditions corresponding to the germ of pointed Frobenius manifold given by the tensor product of $(\mathbf{M}', m_0')$ and $(\mathbf{M}'', m_0'')$, if the base–point of the tensor product is tame.

8.1.2. Remarks.

(i) The dependence on the points relative to which the operation of tensor product is performed, is explicitly described in Section 7.2. In particular, when we pass to a split cover of the subset of tame points, the solution of the Schlesinger equations will be independent of the chosen points due the uniqueness statement of Theorem 3.7.

(ii) In 8.1.4 we will show that it is always possible to perturb the base–points in such a manner that the base–point of the tensor product is indeed tame.

8.1.3. Canonical coordinates. Since the definition of the tensor product, as introduced in Section 7, makes extensive use of the flat coordinates, a natural question to ask in the setting of semi–simple Frobenius manifolds is: Is there a nice formulation in terms of canonical coordinates? Generally, one can not expect simple formulas, since the algebra in the tangent space over a given point in the tensor product manifold is generally not a tensor product of algebras, see 7.2, and the "coupling" of algebras results in a destruction of the pure tensor form of the idempotents.

Using the definitions of the tensor product for formal Frobenius manifolds, we can, however, calculate the idempotents of the tensor product in terms of flat coordinates in the formal situation. They are given by the following Lemma up to terms of order two in flat coordinates which is the precision needed to calculate the special initial conditions.

8.1.4. Lemma. *Given two semi–simple Frobenius manifolds* $\mathbf{M}', \mathbf{M}''$ *let the idempotents near the base–points* m_0', m_0'' *have the expansions* $e_i' = e_i'^0 + \sum x'^{a'} e_i'^{a'} + O(x'^2)$*, and* $e_i'' = e_i''^0 + \sum x''^{a''} e_i''^{a''} + O(x''^2)$ *in the flat coordinates* x' *and* x''*, then the idempotents* e_{ij} *of the tensor product of* $(\mathbf{M}', m_0')$ *and* $(\mathbf{M}'', m_0'')$ *in the formal sense have the following expansion in the flat coordinates* x *of Definition 2.4:*

$$e_{ij}(x) = e_i'^0 \otimes e_j''^0 + \sum_{a',a''} x^{a'a''} (\lambda_j^{a''} e_i'^{a'} \otimes e_j''^0 + \lambda_i^{a'} e_i'^0 \otimes e_j''^{a''}) + O(x^2) \tag{8.1}$$

where $\partial'_{a'} = \sum \lambda_i^{a'} e_i'^0$ *and* $\partial''_{a''} = \sum \lambda_j^{a''} e_j''^0$.

Furthermore, the respective coordinate functions for the tensor metric $\eta_{ij} := \eta(e_{ij}, e_{ij})$ *have the expansions:*

$$\eta_{ij}(x) = \eta_i'(m_0')\eta_j''(m_0'') + \sum_{a',a''} x^{a'a''} (\lambda_j^{a''} (\partial'_{a'}\eta_i')(m_0')\eta_j''(m_0'') + \lambda_i^{a'} \eta_i'(m_0')(\partial''_{a''}\eta_j''(m_0''))) \tag{8.2}$$

and their derivatives $\eta_{ij,kl} := e_{kl}\eta_{ij}$ *have the following values at the base–point* m_0*:*

$$\eta_{ij,kl}(m_0) = \delta_{j,l}\eta_{ik}'(m_0')\eta_j''(m_0'') + \delta_{i,k}\eta_i'(m_0')\partial''_{a''}\eta_{jl}''(m_0'') \tag{8.3}$$

where $\delta_{i,k}$ *is the Kronecker delta symbol.*

The canonical coordinates of m_0 *are:*

$$u^{ij}(m_0) = u'^i(m_0') + u''^j(m_0''). \tag{8.4}$$

Proof. To check the formula (8.1), expand the potential Φ up to order four in the flat coordinates and verify the idempotency by direct calculation. The formulas (8.2) and (8.3) then follow by substitution into the definition of the tensor metric. Finally, (8.4) can be derived from the expansion of the equation $E = \sum u^{ij} e_{ij}$ with the Euler field E given by Theorem 7.1.1. Since the calculation is lengthy, but fairly straightforward, we omit it.

8.2. Tensor product of special initial conditions. In the Lemma 8.1.4, we have calculated all of the structures (3.17) necessary for the determination of the special initial condition. Another approach using the Euler field is given by the following observation:

8.2.1. Remark. To give the special initial conditions for a tensor product with the choice of tensor metric and the Euler field (6.12), it suffices to determine the operator

$$\mathcal{V} : \mathcal{V}(X) = \nabla_{0,X}(E) - \frac{D}{2}X \tag{8.5}$$

in the tangent space to the base–point $\mathcal{T}_{M,m_0}$. Since $\mathcal{V}$ is an $\mathcal{O}_M$–linear tensor, its value on a vector field X is already determined by $X\big|_{m_0} \in \mathcal{T}_{M,m_0}$, so that, if we are only interested in the operator $\mathcal{V}$ restricted to $\mathcal{T}_{M,m_0}$, we can use any extension of the vector $X\mid_{m_0}$ to a vector field in a neighborhood of m_0. Choosing a flat extension X^f, the formula (8.5) simplifies to

$$\mathcal{V}(X)\mid_{m_0}=([X^f,E]-\frac{D}{2}X^f)\mid_{m_0}. \tag{8.6}$$

In particular, in the situation of Theorem 6.3, we can extend the idempotents $e_{ij}\mid_{m_0}$ to flat vector fields $e^f_{i,j}$ and use the formula (8.6) to calculate the special initial conditions via the operator $\mathcal{V}$ for the semi–simple tensor Euler field. Now it is clear that $(e_{ij})\mid_{m_0}=e'_i\mid_{m'_0}\otimes e''_j\mid_{m''_0}$, since the algebra over m_0 is just the tensor of the algebras at the chosen zeros m'_0 and m''_0. Recalling the form of E given by (6.12), we find for flat X,Y

$$[X\otimes Y,E]=[X,E']\otimes Y+X\otimes[Y,E'']-dX\otimes Y. \tag{8.7}$$

Thus,

$$\begin{aligned}\mathcal{V}(e^f_{ij})=[e^f_{i,j},E]-\frac{D}{2}e^f_{i,j}=&\\ &([e'^f_i,E']-\frac{D'}{2}e'^f_i)\otimes e''^f_j+e'^f_i\otimes([e''^f_j,E'']-\frac{D''}{2}e''^f_j).\end{aligned} \tag{8.8}$$

Using the explicit formulas of Lemma 8.1.4 or the Remark 8.2.1, we obtain:

8.2.2. Theorem. *Let $(\mathbf{M}',m'_0)$ and $(\mathbf{M}'',m''_0)$ be two germs of semi–simple Frobenius manifolds with tame base–points which satisfy $u'^i(m'_0)+u''^j(m''_0)\neq u'^k(m'_0)+u''^l(m''_0)$ for $i\neq k$ and $j\neq l$ and let the corresponding special initial conditions be given by $(\mathcal{V}',\eta')$ and $(\mathcal{V}'',\eta'')$, then the special initial conditions for the Schlesinger equations corresponding to the tensor product with the flat identity and the Euler field of the product chosen as in Theorem 6.3 are given by:*

$$\eta_{ij}=\eta'_i\eta'_j \tag{8.9}$$

$$v_{ij,kl}=\delta_{jl}v'_{ik}+\delta_{ik}v''_{jl}.\quad\square \tag{8.10}$$

CHAPTER 3

Quantum Cohomology

As we will explain below, one can regard quantum cohomology as a formal Frobenius manifold structure with Euler field and identity. Taking this viewpoint, we can apply the results of the previous chapter to this situation. In particular, the formula for the tensor product of two formal Frobenius manifolds turns into the explicit Künneth formula for quantum cohomology. Furthermore, if the H^1's of the factors are zero, we find a particularly simple structure for the small quantum cohomology ring using the results on the base–point dependence. In the semi–simple case, we can calculate the special initial conditions for a product manifold and, if the H^1's of the factors are zero, we can calculate the special initial conditions in a first order neighborhood of H^2 if the respective data is given for the factors, again using the results on the base–point dependence.

In the last section we will present some examples. First, we calculate the special initial conditions of a product of projective spaces by combining the results of [M2] with the ones from the previous chapter. And as a final example, we study the product of two and three three–dimensional Calabi–Yau$s$ and give their potentials.

1. Gromov–Witten invariants and quantum cohomology

1.1. GW–invariants. The Gromov–Witten invariants are defined for any smooth projective variety V over $\mathbb{C}$ by means of the space of stable maps. We will not go into details here, since they can be found in [KM, K2, Be1, BM, FP]. We take the viewpoint of [KM] and regard the GW–invariants of a variety V as maps, depending on the parameters g (genus), n (number of points) with $2g-2+n>0$ and the choice of a class $\beta \in H_2(V,\mathbb{Z})$:

$$I^V_{g,n,\beta} : H^*(V)^{\otimes n} \to H^*(\overline{M}_{g,n}). \tag{1.1}$$

These maps satisfy certain properties which were called axioms in [KM].

1.2. GW–numbers. Given a set of GW–invariants, the GW–numbers are defined by

$$\langle I^V_{g,n,\beta}\rangle(\gamma) := \int_{\overline{M}_{g,n}} I^V_{g,n,\beta}(\gamma). \tag{1.2}$$

Using the Second Reconstruction Theorem [KM], it can be shown that the GW–numbers already suffice to define the whole set of GW–classes.

1.3. Quantum Cohomology. Restricting to genus zero invariants, one can encode the information of GW–invariants into an algebra structure. The quantum cohomology of a projective manifold V will be regarded as a formal deformation of its cohomology ring with the coordinates of the space $H^*(V)$ being the parameters. The structure constants are given by a formal series Φ^V which is defined in terms of Gromov–Witten invariants [KM]. One can regard the quantum cohomology as a structure of a Frobenius manifold on $(H^*(V), g)$ where g is the Poincaré pairing with the GW–invariants playing the role of the I_n and the potential Φ^V being the potential of (II.1.20).

1.4. Preparation. Let Λ be a $\mathbb{Q}$–algebra of and choose a character

$$B \to \Lambda; \quad \beta \to q^\beta \tag{1.3}$$

where B is the closure of the cone of effective algebraic curves.

The choices for Λ are basically

(i) $\Lambda = \mathbb{C}, q^\beta = e^{-\int_\beta \omega}$, for a choice of $\omega \in H^2(V, \mathbb{C})$

(ii) $\Lambda =$ the Novikov ring and $q^\beta = q_1^{\beta_1} \dots q_r^{\beta_r}$ for $(\beta = \beta_1, \dots, \beta_r)$.

1.5. CohFT. Summing over all $\beta \in B \subset H_2(V, \mathbb{Z})$, set

$$I_{0n}^V := \sum_\beta q^\beta I_{0,n,\beta}^V \tag{1.4}$$

if (1.4) is convergent as a map

$$H^*(V) \to H^*(\overline{M}_{0n}, \Lambda). \tag{1.5}$$

In this case the I_{0n}^V (1.4) form a CohFT see [KM].

1.6. GW–potential. According to the general framework, we can associate the potential

$$\Phi^V(\gamma) := \sum_{n \geq 3} \frac{1}{n!} \langle I_{0n}^V \rangle (\gamma^{\otimes n}) \tag{1.6}$$

to the above CohFT, cf. 1.3.

1.7. Quantum multiplication. Since the I_{0n}^V form a CohFT, we equivalently have that the above potential satisfies the associativity (WDVV) equations and thus defines a formal Frobenius manifold structure on $(H^*(V), g)$ where g is the Poincaré pairing. This structure is called the big quantum cohomology ring. Explicitly, let (Δ_a) be a basis of $H^*(V)$ and x^a the dual basis. Write out the GW–potential (1.6) for a generic $\gamma = \sum_a x^a \Delta_a = \gamma_0 + \delta$ where $\delta = \sum_{a | \Delta_a \in H^2} x^a \Delta_a$ and set

$$\Delta_a \circ \Delta_b = \sum_c \Phi_{ab}^c \Delta_c \tag{1.7}$$

with

$$\Phi_{ab}^c \in K = \Lambda[[x^a]]. \tag{1.8}$$

1.8. A more explicit form of the potential. Due to the identity and the divisor "axiom" for GW–classes one can give a more precise description of the potential for quantum cohomology.

First, we decompose γ into its H^0, H^2 and $H^i, i \neq 0, 2$ components:

$$\gamma = \gamma_0 + \gamma_2 + \gamma' = x_0[V] + \sum_{i=1}^{r} x_i \Delta_i + \sum_{i=r+1}^{k} y_i \Delta_{r+i} \tag{1.9}$$

where $\Delta_i, i = 1, \dots r$ is a basis for H^2.

Let $\beta_1, \dots \beta_r$ be a dual basis of H_2, then we can write the potential as

$$\Phi^V(\gamma) = \frac{1}{6}(\gamma^3) + \sum_{\beta \in B \setminus \{0\}} q^\beta e^{\beta_1 x_1 + \dots + \beta_r x_r} \Phi_\beta \tag{1.10}$$

where

$$\Phi_\beta = \sum_{n \geq 0} \sum_{(a_1, \dots, a_n)} \frac{y_{a_n} \cdots y_{a_1}}{n!} \langle I^V_{0,n,\beta} \rangle (\Delta_{r+a_1} \otimes \dots \otimes \Delta_{r+a_n}) \tag{1.11}$$

and the $I^V_{0,n,\beta}$ for $n \leq 3$ are defined as

$$I^V_{0,n,\beta}(\alpha) = (\beta, \delta)^{-m} I^V_{0,n+m,\beta}(\alpha \otimes \delta^{\otimes m}) \tag{1.12}$$

for any $\delta \in H^2$ with $(\beta, \delta) \neq 0$.

The summands contributing to Φ_β are constrained by the condition that the virtual dimension should be zero:

$$\sum (|\Delta_{r+a_i}| - 2) = 2(-K_V \cdot \beta + dim_{\mathbb{C}} V - 3). \tag{1.13}$$

In particular, in case $H^1 = 0$ we see that Φ^V is a polynomial in e^{x_i} and y_j, if K_V is ample or V is a Calabi–Yau with $dim \geq 3$. In the latter case, the restriction reads:

$$\sum (|\Delta_{r+a_i}| - 2) = 2(dim_{\mathbb{C}} V - 3). \tag{1.14}$$

On the other hand, the left hand side of (1.14) is greater or equal to n, so that we find the following restriction for n:

$$n \leq 2(dim_{\mathbb{C}} V - 3). \tag{1.15}$$

1.9. Quantum cohomology as a formal pointed Frobenius manifold. As shown above, when we are considering the case of quantum cohomology of a projective manifold V from the vantage point of Frobenius manifolds, the Poincaré pairing and the GW–potential define a formal Frobenius manifold. This formal Frobenius manifold comes equipped with the natural base–point $0 \in H^*(V)$ whose tangent space is identified with $H^*(V)$. We will denote this formal Frobenius manifold by $(H^*_{quant}(V), 0)$.

$(H^*_{quant}(V), 0)$ also has a flat identity e provided by the identity in the usual cohomology ring $1 = [\check{V}]$, the Poincaré dual of the fundamental class, and a semi–simple Euler–field E of weight $D = 2 - \dim(V)$; cf [M2]. Choose a homogeneous basis $\{\Delta_a\}$ of $H^*(V)$ with $\Delta_0 = 1$ and let x^a be the dual coordinates. These coordinates define the flat vector fields ∂_a. The degree d_a of ∂_a being $d_a = 1 - \frac{|\Delta_a|}{2}$ where $|\,.\,|$ is the usual grading of H^*. Let K_V denote the canonical class of V and expand $-K_V = \sum_{b:|\Delta_b|=2} r^b \Delta_b$. Then

$$e = \partial_0 := [\check{V}], \quad E = \sum_a d_a x^a \partial_a + \sum r^b \partial_b. \tag{1.16}$$

1.9.1. The small quantum cohomology ring. Restricting the formal Frobenius manifold to the subspace $H^2 \subset H$, we obtain a multiplication on H with parameters in H^2. To give an explicit formula decompose $\gamma = \gamma_0 + \delta$ where $\delta = \sum_{a|\Delta_a \in H^2} x^a \Delta_a$ and set

$$\Delta_a \circ \Delta_b = \sum_c \Phi_{ab}^c|_{\gamma_0=0} \Delta_c \tag{1.17}$$

with

$$\Phi_{ab}^c \in K = \Lambda[[x^a | \Delta_a \in H^2]]. \tag{1.18}$$

2. The Künneth formula in quantum cohomology

2.1. Quantum cohomology of a product. The quantum cohomology of a product $V \times W$ regarded as a Frobenius manifold is just the tensor product of the formal Frobenius manifolds: $(H^*(V) \otimes H^*(W), \text{Poincaré pairing}, \Phi^{V\times W})$ as can be shown [Be2] using [Be1].

More precisely, in terms of formal pointed Frobenius manifolds

$$(H^*_{quant}(V \times W), 0) = \text{ the tensor product of } (H^*_{quant}(V), 0) \text{ and } (H^*_{quant}(W), 0). \tag{2.1}$$

Furthermore, as already mentioned, the canonical Euler field and the identity for the quantum cohomology of a product coincide with the ones given in Theorem II.6.3.

2.2. The explicit Künneth formula in quantum cohomology. Putting together the formula for the potential (II.1.6) and Corollary II.4.3.3, we obtain the explicit Künneth formula:

2.3. Corollary. *The potential $\Phi^{V\times W}$ of the quantum cohomology of $V \times W$ is given by the formula:*

$$\Phi^{V\times W}(\gamma' \otimes \gamma'') = \sum_{n\geq 3} \frac{1}{n!} \sum_{\mu,\nu\in\mathcal{B}_n} Y'(\check{\mu})(\gamma'^{\otimes n}) m_{\mu\nu} Y''(\check{\nu})(\gamma''^{\otimes n}). \tag{2.2}$$

2.4. The small quantum cohomology of a product. In the case that

$$(*)\ H^1(V) = H^1(W) = 0$$

we can apply the results of Section II.7.2 to find a particularly simple structure for the small quantum cohomology of a product. Recall that the small quantum cohomology ring is the restriction of the deformation space of quantum cohomology to H^2. From the general theory we know that the small quantum cohomology ring of a product is the tensor product of the CohFT, and since $H^2(V \times W, k) = (H^2(V, k) \otimes H^0(W, k)) \oplus (H^0(V, k) \otimes H^2(W, k))$ if $(*)$ holds, we may apply Theorem II.7.2.3. Specializing Corollary II.7.2.5, we obtain:

2.4.1. Corollary. If $H^1(V) = H^1(W) = 0$ then the small quantum cohomology ring of the product $V \times W$ is the tensor product of the small quantum cohomology rings of V and W as k–modules.

2.5. Semi–simplicity and quantum cohomology. Over the canonical base–point 0 of $H^*_{quant}(V)$ the multiplication is always nilpotent, since it is just the usual cohomology with cup product. But if the potential Φ of $H^*_{quant}(V)$ converges in some domain, one can look at different base–points and in some cases the multiplication is generically semi–simple in this domain. This is the case for complete intersections, [TXu]. In this situation, we can change the base–point to a tame semi–simple point and continue the potential analytically, using the formalism of Schlesinger equations.

2.6. Semi–simplicity and the Künneth formula. The Künneth formula (2.1) involves a priori non–semi–simple base–points. If, however, we have convergence of all of the potentials in some big enough domain, we can use the Theorem II.7.2.3 to move the base–points of the factors into tame semi–simple points, if they exist, and then tensor. In fact, we can then even analytically continue after forming the tensor product, if the base–point of the tensor product is tame, using the Schlesinger equations due to the uniqueness statement of Theorem II.3.7.

2.6.1. Special initial conditions in the first order neighborhood of H^2. When considering the special initial conditions given by the quantum cohomology of a manifold, it is usually sufficient to move the base–point only into the space generated by H^2. In order to consider an arbitrary base–point in H^2, one can calculate the relevant structure in the first infinitesimal neighborhood of H^2. Recall that the formal Frobenius structure restricted to H^2 is just the small quantum cohomology.

If we again assume that $H^1(V) = H^1(W) = 0$ then we see that the dependence on $H^0(V \times W)$ and $H^2(V \times W) = (H^0(V) \otimes H^2(W)) \oplus (H^2(V) \otimes H^0(W))$ is given by Theorem II.7.2.3 respectively Corollary II.2.4.1.

As an illustration, we give the formula for the idempotents of the tensor product in a first order neighborhood of H^2 under the condition $H^1(V) = H^1(W) = 0$. Denote the ideal generated by the $x^{ab}|d_{ab} \neq 0, 2$ by J and define J' and J'' analogously, then decompose the idempotents

$$e'_i = e'^0_i + \sum_{a'|x'^{a'} \in J'} x'^{a'} e'^{a'}_i + O(J'^2)$$

and

$$e''_i = e''^0_i + \sum_{a''|x''^{a''} \in J''} x''^{a''} e''^{a''}_i + O(J''^2).$$

Then the idempotents of the tensor product are given by:

$$e_{ij}(x) = e'^0_i \otimes e''^0_j + \sum_{a',a''|x^{a'a''} \in J} x^{a'a''} (\lambda^{a''}_j e'^{a'}_i \otimes e''^0_j + \lambda^{a'}_i e'^0_i \otimes e''^{a''}_j) + O(J^2). \tag{2.3}$$

3. Examples

3.1. Special initial conditions for $\mathbb{P}^n \times \mathbb{P}^m$. Using Section II.8.2, we can calculate the special initial conditions for $\mathbb{P}^n \times \mathbb{P}^m$ using the results of [MM]. Set $\zeta_n = \exp(\frac{2\pi i}{n+1})$.

3.1.1. Proposition. *The point* $(x^{00}, x^{10}, x^{01}, 0, \dots)$ *has canonical coordinates* $u_{ij} = x^{00} + \zeta_n^i (n+1) e^{\frac{x^{10}}{n+1}} + \zeta_m^i (m+1) e^{\frac{x^{01}}{m+1}}$

The special initial conditions at this point corresponding to $H_{quant}(\mathbb{P}^n \times \mathbb{P}^m)$ *are given by*

$$v_{ij,kl} = -\left(\frac{\zeta_n^{i-k}}{1-\zeta_n^{i-k}}\delta_{jl} + \frac{\zeta_m^{j-l}}{1-\zeta_m^{j-l}}\delta_{ik}\right) \tag{3.1}$$

and

$$\eta_{ij} = \frac{\zeta_n^i \zeta_m^j}{(n+1)(m+1)} e^{-x^{10}\frac{n}{n+1} - x^{01}\frac{m}{m+1}} \tag{3.2}$$

Proof. Taking the results of [MM], we can use Theorem II.7.2.3 to move the base–point in an appropriate way and then apply Theorem II.8.2.2 to calculate the special initial conditions. The formula for the canonical coordinates is contained in Lemma II.8.1.4 and the fact that in the presence of an Euler field $x^{00} = x'^0 + x''^0$, is again given by Theorem II.7.2.3.

3.2. The potential for a product of Calabi–Yau*s*. From (1.15) we see that, if we are considering a Calabi–Yau manifold which is a product of two Calabi–Yau manifolds, we only need to consider a finite number of n's in order to calculate the potential. Thus, if the potentials of the factors are known, we can use the explicit computations of the previous chapter in order to calculate the potential of the product.

3.2.1. Restrictions for a product of Calabi–Yau*s*. In the case of a product of Calabi–Yau*s*, there are additional restrictions besides the restriction (1.13) for the Calabi–Yau W on the total degree of the cohomology classes. We choose the basis (Δ_i) of $H^*(W)$ in such a way that $\Delta_i \in H^j(V_1) \otimes H^k(V_2), k+l = |\Delta_i| := d_i$ and consider the bidegrees $(d_i^1, d_i^2) := (k, l)$.

Since $K_V = 0$, we find that Φ is homogeneous and the homogeneity condition for the naive tensor product of abstract correlation functions for a bihomogeneous element $\sigma \otimes \tau \in V(\Gamma_{n,e_1}) \otimes V(\Gamma_{n,e_2})$ —where we used the notation of Chapter I.3.1— reads:

$$(Y_1 \otimes Y_2)(\sigma \otimes \tau)(\Delta_{i_1} \otimes \dots \otimes \Delta_{i_n}) = 0$$

unless

$$\sum_j (d_{i_j}^1 - 2) = 2(\dim_{\mathbb{C}} V_1 - 3 - e_1) \text{ and } \sum_j (d_{i_j}^2 - 2) = 2(\dim_{\mathbb{C}} V_2 - 3 - e_2). \tag{3.3}$$

3.2.2. The product of two three–dimensional Calabi–Yau*s*. As an example of a higher dimensional Calabi–Yau we will consider a product of two three–dimensional Calabi–Yau*s* $W = V_1 \times V_2$ with $H^2(V_1) = H^2(V_0) = 1$ for simplicity. In this situation H_2 is generated by $\pi_1^*(\beta_1)$ and $\pi_2^*(\beta_2)$ and $B = \{(d_1, d_2) | d_1, d_2 \geq 0\}$. We will consider the basis formed by the elements $\Delta_{ij} = \Delta_i^{(1)} \otimes \Delta_j^{(2)}$ and denote the respective dual coordinates by x_{01}, x_{10} and y_{ij} for $i + j > 2$. Let $k_i := \dim H^*(V_i)$.

Using the formula for the potential (1.10) and the restriction (1.15) we find for V_1 and V_2:

$$\Phi^{V_i}(\gamma) = \frac{1}{6}(\gamma^3) + \sum_{\substack{d \geq 1 \\ \beta = d\, c_1(\mathcal{O}(1))}} q^d e^{d x_1^i} \langle I_{0,0,\beta}^{V_i} \rangle (*). \tag{3.4}$$

where $(*)$ denotes the empty product and

$$\langle I_{0,0,\beta}^{V_i} \rangle (*) := \frac{\langle I_{0,3,d\, c_1(\mathcal{O}(1))}^{V_i} \rangle (\Delta_1^{\otimes 3})}{d^3} = \frac{N_i(d)}{d^3} \tag{3.5}$$

with $N_i(d) = \langle I_{0,3,d\, c_1(\mathcal{O}(1))}^{V_i} \rangle$ for all $d \geq 0$.

First, we list the summands of Φ_β which may contribute. Using the restriction (1.13) we find that independent of β, $n \leq 6$. The following tuples of d_{a_i}'s might lead to nonzero contributions:

$$\begin{aligned} \{d_{a_1}, \ldots, d_{a_n}\} \in \{&\{8\}, \{3,7\}, \{4,6\}, \{5,5\}, \{3,3,6\}, \{3,4,5\}, \{4,4,4\} \\ &\{3,3,3,5\}, \{3,3,4,4\}, \{3,3,3,3,4\}\{3,3,3,3,3,3\}\}. \end{aligned} \tag{3.6}$$

Furthermore, we have the conditions on the bidegree imposed by the homogeneity ($K_V = 0$) condition for Φ^W which are given by:

$$\sum_j d_{i_j}^1 = 2(n - e_1) \text{ and } \sum_j d_{i_j}^2 = 2(n - e_2). \tag{3.7}$$

For $n \leq 3$ we find the following table of bidegrees where we have expanded the cases with $n \leq 2$, and possible β for which $\langle I_{0,3,\beta}^W \rangle (\Delta_{a_1} \otimes \Delta_{a_2} \otimes \Delta_{a_3})$ is not necessarily zero:

Let $\{i,j,k\} = \{1,2,3\}$

$d_{a_i}, d_{a_j}, d_{a_k}$	$d^1_{a_i}, d^1_{a_j}, d^1_{a_k}$	$d^2_{a_i}, d^2_{a_j}, d^2_{a_k}$	β
$2,2,8$	$0,2,4$	$2,0,4$	0
	$0,0,6$	$2,2,2$	$(0,d)$
	$2,2,2$	$0,0,6$	$(d,0)$
$2,3,7$	$0,3,3$	$2,0,4$	0
	$2,0,4$	$0,3,3$	0
$2,4,6$	$2,4,0$	$0,0,6$	0
	$0,0,6$	$2,4,0$	0
	$2,0,4$	$0,4,2$	0
	$0,4,2$	$2,0,4$	0
	$0,2,4$	$2,2,2$	$(0,d)$
	$2,2,2$	$0,2,4$	$(d,0)$
$2,5,5$	$0,3,3$	$2,2,2$	$(0,d)$
	$2,2,2$	$0,3,3$	$(d,0)$
$3,3,6$	$0,0,6$	$3,3,0$	0
	$3,3,0$	$0,0,6$	0
	$0,3,3$	$3,0,3$	0
$3,4,5$	$3,0,3$	$0,4,2$	0
	$0,4,2$	$3,0,3$	0
$4,4,4$	$2,4,0$	$2,0,4$	0
	$2,2,2$	$2,2,2$	(d_1,d_2)

The cases where $\beta = 0$ is the only possibility for a non–zero contribution are not relevant, since we have the condition $\beta \neq 0$ in the sum over the Φ_β. These cases are only listed for completeness and future reference.

In the case $n = 4$ and $\{i,j,k,l\} = \{1,\dots,4\}$ the possible list reads:

$d_{a_i}, d_{a_j}, d_{a_k}, d_{a_l}$	$d^1_{a_i}, d^1_{a_j}, d^1_{a_k}, d^1_{a_l}$	$d^2_{a_i}, d^2_{a_j}, d^2_{a_k}, d^2_{a_l}$	e_1, e_2
$3,3,3,5$	$0,3,3,2$	$3,0,0,3$	$0,1$
	$3,0,0,3$	$0,3,3,2$	$1,0$
$3,3,4,4$	$3,3,0,0$	$0,0,4,4$	$0,1$
	$0,0,4,4$	$3,3,0,0$	$1,0$
	$0,0,2,4$	$3,3,2,0$	$0,1$
	$3,3,2,0$	$0,0,2,4$	$1,0$

All these cases yield zero contributions due to Proposition 6.2.2, since $\pi_*(H^0(\overline{M}_{0,4})) = 0$.

In the case $n = 5$ we get

$d_{a_i}, d_{a_j}, d_{a_k}, d_{a_l}, d_{a_m}$	$d^1_{a_i}, d^1_{a_j}, d^1_{a_k}, d^1_{a_l}, d^1_{a_m}$	$d^2_{a_i}, d^2_{a_j}, d^2_{a_k}, d^2_{a_l}, d^2_{a_m}$	e_1, e_2
$3,3,3,3,4$	$0,0,3,3,4$	$3,3,0,0,0$	$0,2$
	$0,0,3,3,2$	$3,3,0,0,2$	$1,1$
	$3,3,0,0,0$	$0,0,3,3,4$	$2,0$

All these cases also only yield zero contributions, since $\pi_*(H^0(\overline{M}_{0,5})) = 0$ and $\pi_{\{i,j\}*}(H^1(\overline{M}_{0,5})) = 0$.

And finally for $n=6$ the possible bidegrees for $\{d_{a_i}, d_{a_j}, d_{a_k}, d_{a_l}, d_{a_m}, d_{a_p}\} = \{3,3,3,3,3,3\}$ are:

$d^1_{a_i}, d^1_{a_j}, d^1_{a_k}, d^1_{a_l}, d^1_{a_m}, d^1_{a_p}$	$d^2_{a_i}, d^2_{a_j}, d^2_{a_k}, d^2_{a_l}, d^2_{a_m}, d^2_{a_p}$	e_1, e_2
$0,0,3,3,3,3$	$3,3,0,0,0,0$	$0,3$
$3,3,0,0,0,0$	$0,0,3,3,3,3$	$3,0$

which also just give zero contributions, since $\pi_*(H^0(\overline{M}_{0,5}))=0$.

Collecting the results, we see that there are only contributions for $n \leq 3$ and with the notation introduced above the potential reads:

$$\begin{aligned}\Phi^W(\gamma) =& \frac{1}{6}(\gamma^3) + \sum_{\substack{d\geq 1\\ \beta=(d,0)}} [\frac{1}{6} y_{1,k_2}\frac{N_1(d)}{d^2} + \frac{1}{3} y_{1,1}y_{1,k_2-1}\frac{N_1(d)}{d} \\ &+ \sum_{(i,j)\in\{2,\ldots,k_2-1\}} \frac{1}{6} y_{1,j}y_{1,i}g^{(2)}_{ij}\frac{N_2(d)}{d} + \frac{1}{6}y^3_{1,1}N_1(d)]q^{(0,d)}e^{dx_{1,0}} \\ &+ \sum_{\substack{d\geq 1\\ \beta=(0,d)}} [\frac{1}{6} y_{k_1,1}\frac{N_2(d)}{d^2} + \frac{1}{3} y_{1,1}y_{k_1-1,1}\frac{N_2(d)}{d} \\ &+ \sum_{(i,j)\in\{2,\ldots,k_1-1\}} \frac{1}{6} y_{j,1}y_{i,1}g^{(1)}_{ij}\frac{N_1(d)}{d} + \frac{1}{6}y^3_{1,1}N_2(d)]q^{(d,0)}e^{dx_{0,1}} \\ &+ \sum_{\substack{d_1,d_2\geq 0,(d_1,d_2)\neq(0,0)\\ \beta=(d_1,d_2)}} \frac{1}{6}y^3_{1,1}N_1(d_1)N_2(d_2)q^{(d_1,d_2)}e^{d_1x_{1,0}+d_2x_{0,1}}. \end{aligned} \tag{3.8}$$

3.2.3. Remark. Regarding (3.8) we notice that the quantum multiplication is still only dependent on H^2 and is thus completely determined by the small quantum cohomology ring. In comparison to the case of a single three–dimensional Calabi–Yau, a new feature emerges, however. Although there are no deformation parameters outside H^2, there are instanton or quantum corrections to the multiplication of odd dimensional classes. Of course, there are no such corrections in the case of classes from H^3, since due to the fact that $H^1=0$ the dependence on these classes is already determined by Corollary II.7.2.5.

3.3. The product of three three–dimensional Calabi–Yau*s*. Since there were no contributions in the last example for $n>3$ and therefore no difference between the information carried by the small and the big quantum cohomology ring appears, we will consider the product of three three–dimensional Calabi–Yau*s* $W=((V\times V)\times V)$ with the same conditions as above $H^2(V)=1$ for simplicity. The basis of $H^*(W)$ is again chosen to be the triple tensor product of the basis for $H^*(V)$ with dual coordinates $x_{100}, x_{010}, x_{001}$ and y_{ijl} for $i+j+l>2$ and $B=\{(d_1,d_2,d_3\}|d_i\geq 0\}$. As indicated by the parentheses, we will view W as the product of a six–dimensional Calabi–Yau $U=V\times V$ whose GW–numbers were calculated above and a three–dimensional Calabi–Yau V.

3.3.1. The classes $I^W_{0,3,\beta}$. From the general theory we see that

$$\langle I^W_{0,3,\beta=(d_1,d_2,d_3)}\rangle(\gamma_{i_1,j_1,k_1}\otimes\gamma_{i_2,j_2,k_2}\otimes\gamma_{i_3,j_3,k_3}) = \langle I^V_{0,3,d_1}\rangle(\gamma_{i_1}\otimes\gamma_{j_1}\otimes\gamma_{k_1})\langle I^V_{0,3,d_2}\rangle(\gamma_{i_2}\otimes\gamma_{j_2}\otimes\gamma_{k_2})\langle I^V_{0,3,d_3}\rangle(\gamma_{i_3}\otimes\gamma_{j_3}\otimes\gamma_{k_3}) \quad (3.9)$$

and the list for the $\langle I^V_{0,3,d}\rangle$ (up to permutations) is simply:

$d_{i_1}, d_{i_2}, d_{i_3}$	β	$\langle I^V_{0,3,\beta}\rangle(\gamma_{i_1}\otimes\gamma_{i_2}\otimes\gamma_{i_3})$
$0,0,6$	0	1
$0,2,4$	0	1
$0,3,3$	0	g_{i_2,i_3}
$2,2,2$	0	1
$2,2,2$	$d\,c_1(\mathcal{O}(1)), d>0$	$N(d)$

Inserting the table into (3.9), we get the contributions to Φ_β with $n\le 3$. Note that the condition $\beta\neq 0$ implies that at least one of the triples has to be $2,2,2$. The final list is rather long and is obtained by simple combinatorics; thus we omit it. The list for only two factors was given in the previous paragraph.

Since the restrictions for U and V provide a rather lengthy list of possibilities for $n>3$, we work out an additional restriction.

3.3.2. Claim. *Let γ be a class in $H^3(W)$ then*

$$I^W_{0,n,\beta}(\gamma\otimes\alpha)=0 \quad \forall n>3. \tag{3.10}$$

Proof. First we remark that since $H^1(V)=H^1(U)=0$: $H^3(W)=H^0(U)\otimes H^3(V)\oplus H^3(U)\otimes H^0(V)$ and we decompose $\gamma=\gamma'\otimes\Delta_0+\Delta_0\otimes\gamma''$ accordingly. From the previous calculation we know that for $\gamma'\in H^3(U)$

$$I^U_{0,n,\beta}(\gamma'\otimes\alpha)=0 \quad \forall n>3 \tag{3.11}$$

and by (1.13) for $\gamma''\in H^3(V)$

$$I^V_{0,n,\beta}(\gamma''\otimes\alpha)=0 \quad \forall n>3 \tag{3.12}$$

as well. Now for $n>3$ using the results of the last chapter we find:

$$\begin{aligned} I^W_{0,n,\beta}(\gamma\otimes\alpha) = {}& (Y'\otimes Y'')(\Delta_{\overline{M}_{0n}})(\gamma'\otimes\Delta_0\otimes\alpha)+(Y'\otimes Y'')(\Delta_{\overline{M}_{0n}})(\Delta_0\otimes\gamma''\otimes\alpha)\\ = {}&(Y'\otimes Y'')(\pi_1^*,id)(\Delta_{\overline{M}_{0,\{2,\dots,n\}}})(\gamma'\otimes\alpha)+(Y'\otimes Y'')(id,\pi_1^*)(\Delta_{\overline{M}_{0,\{2,\dots,n\}}})(\gamma''\otimes\alpha) \end{aligned} \tag{3.13}$$

By the previous remarks (3.11) and (3.12) $Y'(\tau)(\gamma'\otimes\alpha)$ and $Y''(\tau)(\gamma''\otimes\alpha)$ can only be nonzero, if γ' or respectively γ'' is assigned to a three–valent vertex by (3.11) or (3.12). On the other hand, due to stability no summand of the pull–back $\tau=\pi_s^*(\sigma)$ contains a three–valent vertex with the flag s emanating from it. Thus we find that $I^W_{0,n,\beta}(\gamma\otimes\alpha)=0$ which proves the claim. Alternatively we could have also used Corollary II.7.2.5.

3.3.3. The possible list of degrees for $n > 3$. If we use all restrictions, we see that we should not have any classes of degrees $1, 2, 3$ for $n > 3$ and $n \leq 12$. Putting everything together, we obtain the following list of possible degrees in Φ^W_β for $n > 3$ with $\sum_i^n d_{ai} = 12 + 2n$:

$$\{d_{a_1}, \ldots, d_{a_n}\} = \{\{4,4,4,8\}, \{4,4,5,7\}, \{4,4,6,6\}, \{4,5,5,6\}, \{5,5,5,5\}, \\ \{4,4,4,4,6\}, \{4,4,4,5,5\}, \{4,4,4,4,4,4\}\}. \quad (3.14)$$

Looking at (3.14), we see that effectively we only have to deal with $n \leq 6$. As in the previous example we will tabulate the possible bidegrees. The conditions on the bidegrees are given by:

$$\sum_j d^1_{i_j} = 2(n + 3 - e_1) \qquad \text{and} \qquad \sum_j d^2_{i_j} = 2(n - e_2) \qquad (3.15)$$

From the previous calculation we know that for $n > 3$ the $I^U_{0,n,\beta}$ are not zero for degrees

$$\{d^1_{a_1}, \ldots, d^1_{a_n}\} \in \{\{2, \ldots, 2, 8\}, \{2, \ldots, 2, 4, 6\}, \{2, \ldots, 2, 5, 5\}, \{2, \ldots, 2, 4, 4, 4\}\} \quad (3.16)$$

and the $I^V_{0,n,\beta}$ are not zero only for degrees

$$\{d^2_{a_1}, \ldots, d^2_{a_n}\} = \{2, \ldots, 2\}. \qquad (3.17)$$

A last restriction is given by the inequality for the degrees of the classes in $H^*(V)$: $\#\{\gamma'' | \deg(\gamma'') = 0\} \leq e_2$ for $e_2 < n - 3$. This is due to the property that $Y(\tau)(\alpha \otimes \Delta_0) = Y(\pi_*(\tau))(\alpha)$ for any τ with more than three tails. Recalling that $2(n - e_2) = \sum \deg(\gamma'')$, we see that the only possibility for the degrees of the arguments of $I^V_{0,n,\beta}(\tau)$ with $|E_\tau| = e < n - 3$ is e_2 times zero and $n - e_2$ times two.

3.3.4. Claim. For V as above and any τ in $V(\Gamma_{n,e})$, $e < n - 3$:

$$I^V_{0,n,\beta}(\tau)(\gamma_1 \otimes \cdots \otimes \gamma_n) \neq 0 \qquad (3.18)$$

only if

$$\{|\gamma_1|, \ldots, |\gamma_n|\} = \{\overbrace{0, \ldots, 0}^{e \text{ times}}, \overbrace{2, \ldots, 2}^{n-e \text{ times}}\}. \qquad (3.19)$$

3.3.5. The tables for $n = 4, 5, 6$. Since the formula for the whole potential is very long we will also list the contribution to Φ_β with each case.

In the case $n = 4$, we have the following possibilities:

d_{a_i}	$d^1_{a_i}$	$d^2_{a_i}$	e_1	e_2	β	contribution to Φ_β
$4,4,4,8$	$2,2,2,8$	$2,2,2,0$	0	1	$(d_1,0,d_3)$	$\frac{1}{6}y_{101}^3 y_{1k0} d_1 N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{6}y_{011}^3 y_{k10} d_2 N(d_2)N(d_3)$
	$2,2,4,6$	$2,2,0,2$	0	1	$(d_1,0,d_3)$	$\frac{1}{2}y_{101}y_{110}^2 y_{1k-11} d_1 N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{2}y_{011}y_{110}^2 y_{1k-11} d_2 N(d_2)N(d_3)$
	$2,4,4,4$	$2,0,0,4$	0	1	$(d_1,d_2,0)$	$\frac{1}{2}(d_1 y_{101} + d_2 y_{011})y_{110}^2 y_{11k-1} N(d_1)N(d_2)$
	$4,4,4,2$	$0,0,0,6$	0	1	$(d_1,d_2,0)$	$\frac{1}{6}(d_1 y_{10k} + d_2 y_{01k})y_{110}^3 N(d_1)N(d_2)$
	$2,2,2,6$	$2,2,2,2$	1	0	$(d_1,0,d_3)$	$d_3(\frac{1}{6}y_{101}^3 y_{0k1} + \frac{1}{2}y_{101}^2 y_{011} y_{1k-11})N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$d_3(\frac{1}{6}y_{011}^3 y_{k01} + \frac{1}{2}y_{011}^2 y_{101} y_{k-111})N(d_2)N(d_3)$
$4,4,5,7$	$2,2,5,5$	$2,2,0,2$	0	1	$(d_1,0,d_3)$	$\frac{1}{2}d_1 \sum_{i,j\in\{2,\dots k-2\}} y_{101}^2 y_{1i0} y_{1j0} g_{ji} N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{2}d_2 \sum_{i,j\in\{2,\dots k-2\}} y_{011}^2 y_{i10} y_{j10} g_{ji} N(d_2)N(d_3)$
	$4,4,2,4$	$0,0,3,3$	0	1	$(d_1,d_2,0)$	$\frac{1}{2}\sum_{i,j\in\{2,\dots k-2\}}(d_1 y_{10i} + d_2 y_{01i})y_{110}^2 y_{11j} g_{ji}$ $N(d_1)N(d_2)$
	$2,2,3,5$	$2,2,2,2$	1	0	$(d_1,0,d_3)$	$\frac{1}{2}d_3 \sum_{i,j\in\{2,\dots k-2\}} y_{101}^2 y_{0i1} y_{1j1} g_{ji} N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{2}d_3 \sum_{i,j\in\{2,\dots k-2\}} y_{011}^2 y_{i01} y_{j11} g_{ji} N(d_2)N(d_3)$
$4,4,6,6$	$2,4,6,2$	$2,0,0,4$	0	1	$(d_1,0,0)$	$d_1 y_{101} y_{110} y_{1k-10} y_{10k-1} N(d_1)$
					$(0,d_2,0)$	$d_2 y_{011} y_{110} y_{k-110} y_{01k-1} N(d_2)$
	$2,2,4,6$	$2,2,2,0$	0	1	$(d_1,0,d_3)$	$\frac{1}{2}d_1 y_{101}^2 y_{111} y_{1k-10} N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{2}d_2 y_{011}^2 y_{111} y_{k-110} N(d_2)N(d_3)$
	$4,4,4,2$	$0,0,2,4$	0	1	$(d_1,d_2,0)$	$\frac{1}{2}(d_1 y_{10k-1} + d_2 y_{01k-1})y_{110}^2 y_{111} N(d_1)N(d_2)$
	$2,4,4,4$	$2,0,2,2$	0	1	(d_1,d_2,d_3)	$\frac{1}{2}(d_1 y_{101} + d_2 y_{011})y_{110} y_{111}^2 N(d_1)N(d_2)N(d_3)$
	$2,2,4,4$	$2,2,2,2$	1	0	$(0,0,d_3)$	$d_3 y_{101} y_{011} y_{0k-11} y_{k-101} N(d_3)$
					$(d_1,0,d_3)$	$\frac{1}{2}d_3 y_{101}^2 y_{111} y_{0k-11} N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{2}d_3 y_{011}^2 y_{111} y_{k-101} N(d_2)N(d_3)$
					(d_1,d_2,d_3)	$\frac{1}{2}d_3 y_{101} y_{010} y_{111}^2 N(d_1)N(d_2)N(d_3)$
$4,5,5,6$	$4,2,2,6$	$0,3,3,0$	0	1	$(d_1,0,0)$	$d_1 \sum_{i,j\in\{2,\dots k-2\}} y_{10i} y_{110} y_{10j} y_{1k-10} g_{ji} N(d_1)$
					$(0,d_2,0)$	$d_2 \sum_{i,j\in\{2,\dots k-2\}} y_{01i} y_{110} y_{01j} y_{k-110} g_{ji} N(d_2)$
	$2,5,5,2$	$2,0,0,4$	0	1	$(d_1,0,0)$	$d_1 \sum_{i,j\in\{2,\dots k-2\}} y_{101} y_{1i0} y_{1j0} y_{10k-1} g_{ji} N(d_1)$
					$(0,d_2,0)$	$d_2 \sum_{i,j\in\{2,\dots k-2\}} y_{011} y_{i10} y_{j10} y_{01k-1} g_{ji} N(d_2)$
	$2,3,3,4$	$2,2,2,2$	1	0	$(0,0,d_3)$	$d_3 \sum_{i,j\in\{2,\dots k-2\}} y_{i01} y_{j01} y_{011} y_{0k-11} g_{ji} N(d_3)+$ $d_3 \sum_{i,j\in\{2,\dots k-2\}} y_{0i1} y_{0j1} y_{101} y_{k-101} g_{ji} N(d_3)$
$5,5,5,5$	$2,2,5,5$	$3,3,0,0$	0	1	$(d_1,0,0)$	$d_1 \sum_{i,j,r,s\in\{2,\dots k-2\}} y_{10i} y_{10j} y_{1r0} y_{1s0} g_{ji} g_{sr} N(d_1)$
					$(0,d_2,0)$	$d_2 \sum_{i,j,r,s\in\{2,\dots k-2\}} y_{01i} y_{01j} y_{r10} y_{s10} g_{ji} g_{sr} N(d_2)$
	$3,3,3,3$	$2,2,2,2$	1	0	$(0,0,d_3)$	$d_3 \sum_{i,j,r,s\in\{2,\dots k-2\}} y_{i01} y_{j01} y_{0r1} y_{0s1} g_{ji} g_{sr} N(d_3)$

In the case $n = 5$, we obtain the following table in which we omitted the summations of the type $\sum_{i,j\in\{2,\dots k-2\}}$ due to lack of space.

d_{a_i}	$d^1_{a_i}$	$d^2_{a_i}$	e_1	e_2	β	contribution to Φ_β
$4,4,4,4,6$	$2,2,2,4,6$	$2,2,2,0,0$	0	2	$(d_1,0,d_3)$	$\frac{1}{6}d_1^2y_{101}^3y_{110}y_{1k-10}N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{6}d_2^2y_{011}^3y_{110}y_{k-110}N(d_2)N(d_3)$
	$2,2,4,4,4$	$2,2,0,0,2$	0	2	(d_1,d_2,d_3)	$\frac{1}{2}(\frac{1}{2}d_1^2y_{101}^2+d_1d_2y_{101}y_{010}+\frac{1}{2}d_2^2y_{010}^2)$ $\times y_{110}^2y_{111}N(d_1)N(d_2)N(d_3)$
	$4,4,4,2,2$	$0,0,0,2,4$	0	2	$(d_1,d_2,0)$	$\frac{1}{6}(d_1^2y_{101}y_{10k-1}+d_2^2y_{011}y_{01k-1}+$ $+d_1d_2(y_{101}y_{01k-1}+y_{011}y_{10k-1}))$ $\times y_{110}^3N(d_1)N(d_2)$
	$4,2,2,2,4$	$0,2,2,2,2$	1	1	(d_1,d_2,d_3)	$\frac{1}{2}d_3(d_1y_{101}+d_2y_{011})y_{101}y_{110}y_{111}y_{011}$ $\times N(d_1)N(d_2)N(d_3)$
					$(d_1,0,d_3)$	$\frac{1}{6}d_1d_3y_{101}^3y_{110}y_{0k-11}N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{6}d_2d_3y_{011}^3y_{110}y_{k-101}N(d_2)N(d_3)$
	$2,2,2,2,6$	$2,2,2,2,0$	1	1	$(d_1,0,d_3)$	$\frac{1}{6}d_1y_{101}^3y_{1k-10}y_{011}d_3N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{6}d_2y_{011}^3y_{k-110}y_{101}d_3N(d_2)N(d_3)$
	$2,2,2,2,4$	$2,2,2,2,2$	2	0	$(d_1,0,d_3)$	$\frac{1}{6}y_{101}^3y_{011}y_{0k-11}d_3^2N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{6}y_{101}^3y_{011}y_{0k-11}d_3^2N(d_2)N(d_3)$
					(d_1,d_2,d_3)	$\frac{1}{4}y_{101}^2y_{011}^2y_{111}d_3^2N(d_1)N(d_2)N(d_3)$
$4,4,4,5,5$	$2,2,2,5,5$	$2,2,2,0,0$	0	2	$(d_1,0,d_3)$	$\frac{1}{6}d_1^2y_{101}^3y_{1i0}y_{1j0}g_{ji}N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{6}d_2^2y_{011}^3y_{i10}y_{j10}g_{ji}N(d_2)N(d_3)$
	$4,4,4,3,3$	$0,0,0,2,2$	0	2	$(d_1,d_2,0)$	$\frac{1}{6}(d_1^2y_{10i}y_{10j}+d_1d_2y_{10i}y_{01j}+$ $+d_2^2y_{01i}y_{01j})y_{110}^3g_{ji}N(d_1)N(d_2)$
	$2,2,2,3,5$	$2,2,2,2,0$	1	1	$(d_1,0,d_3)$	$\frac{1}{6}d_1y_{101}^3y_{1i1}y_{1j1}g_{ji}d_3N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{6}d_2y_{011}^3y_{i11}y_{j11}g_{ji}d_3N(d_2)N(d_3)$
	$4,2,2,3,3$	$0,2,2,2,2$	1	1		0
	$2,2,2,3,3$	$2,2,2,2,2$	2	0	$(d_1,0,d_3)$	$\frac{1}{6}d_3^2y_{101}^3y_{1i1}y_{1j1}g_{ji}N(d_1)N(d_3)$
					$(0,d_2,d_3)$	$\frac{1}{6}d_3^2y_{011}^3y_{i11}y_{j11}g_{ji}N(d_2)N(d_3)$

The last case $n = 6$ renders $\{d_{a_i}\} = \{4,4,4,4,4,4\}$:

$d^1_{a_i}$	$d^2_{a_i}$	e_1	e_2	β	contribution to Φ_β
$2,2,2,4,4,4$	$2,2,2,0,0,0$	0	3	(d_1,d_2,d_3)	$\frac{1}{6}(\frac{1}{6}d_1^3y_{101}^3+\frac{1}{2}d_1^2d_2y_{101}^2y_{011}+\frac{1}{2}d_1d_2^2y_{101}y_{011}^2+$ $+\frac{1}{6}d_2^3y_{011}^3)y_{110}^3N(d_1)N(d_2)N(d_3)$
$2,2,2,2,4,4$	$2,2,2,2,0,0$	1	2	(d_1,d_2,d_3)	$\frac{1}{2}(\frac{1}{6}d_1^2y_{101}^3y_{011}+\frac{1}{4}2d_1d_2y_{101}^2y_{011}^2+$ $+\frac{1}{6}d_2^2y_{101}y_{011}^3)y_{110}^2d_3N(d_1)N(d_2)N(d_3)$
$2,2,2,2,2,4$	$2,2,2,2,2,0$	2	1	(d_1,d_2,d_3)	$\frac{1}{12}(d_1y_{101}+d_2y_{011})y_{101}^2y_{011}^2y_{110}$ $\times d_3^2N(d_1)N(d_2)N(d_3)$
$2,2,2,2,2,2$	$2,2,2,2,2,2$	3	0	(d_1,d_2,d_3)	$\frac{1}{36}y_{101}^3y_{011}^3d_3^3N(d_1)N(d_2)N(d_3)$

Summing up all $n=4$ contributions:

β	contribution to Φ_β
$(d_1,0,0)$	$d_1 y_{101}y_{110}y_{1k-10}y_{10k-1}N(d_1)+$
	$d_1\sum_{i,j\in\{2,\dots k-2\}} y_{10i}y_{110}y_{10j}y_{1k-10}g_{ji}N(d_1)+$
	$d_1\sum_{i,j\in\{2,\dots k-2\}} y_{101}y_{1i0}y_{1j0}y_{10k-1}g_{ji}N(d_1)+$
	$d_1\sum_{i,j,r,s\in\{2,\dots k-2\}} y_{10i}y_{10j}y_{1r0}y_{1s0}g_{ji}g_{sr}N(d_1)$
$(0,d_2,0)$	$d_2 y_{011}y_{110}y_{k-110}y_{01k-1}N(d_2)+$
	$d_2\sum_{i,j\in\{2,\dots k-2\}} y_{01i}y_{110}y_{01j}y_{k-110}g_{ji}N(d_2)+$
	$d_2\sum_{i,j\in\{2,\dots k-2\}} y_{011}y_{i10}y_{j10}y_{01k-1}g_{ji}N(d_2)+$
	$d_2\sum_{i,j,r,s\in\{2,\dots k-2\}} y_{01i}y_{01j}y_{r10}y_{s10}g_{ji}g_{sr}N(d_2)$
$(0,0,d_3)$	$d_3 y_{101}y_{011}y_{0k-11}y_{k-101}N(d_3)+$
	$d_3\sum_{i,j\in\{2,\dots k-2\}} y_{i01}y_{j01}y_{011}y_{0k-11}N(d_3)+$
	$d_3\sum_{i,j\in\{2,\dots k-2\}} y_{0i1}y_{0j1}y_{101}y_{k-101}g_{ji}N(d_3)+$
	$d_3\sum_{i,j,r,s\in\{2,\dots k-2\}} y_{i01}y_{j01}y_{0r1}y_{0s1}g_{ji}g_{sr}N(d_3)$
$(d_1,d_2,0)$	$\frac{1}{6}d_1 y_{10k}y_{110}^3N(d_1)N(d_2)+$
	$\frac{1}{2}d_1 y_{101}y_{110}^2y_{11k-1}N(d_1)N(d_2)+$
	$\frac{1}{2}d_1 y_{10k-1}y_{110}^2y_{111}N(d_1)N(d_2)+$
	$\frac{1}{6}d_2 y_{01k}y_{110}^3N(d_1)N(d_2)+$
	$\frac{1}{2}d_2 y_{011}y_{110}^2y_{11k-1}N(d_1)N(d_2)+$
	$\frac{1}{2}d_2 y_{01k-1}y_{110}^2y_{111}N(d_1)N(d_2)+$
	$\frac{1}{2}\sum_{i,j\in\{2,\dots k-2\}} d_1 y_{10i}y_{110}^2y_{11j}g_{ji}N(d_1)N(d_2)+$
	$\frac{1}{2}\sum_{i,j\in\{2,\dots k-2\}} d_2 y_{01i}y_{110}^2y_{11j}g_{ji}N(d_1)N(d_2)$
$(d_1,0,d_3)$	$\frac{1}{6}d_1 y_{101}^3y_{1k0}N(d_1)N(d_3)+$
	$\frac{1}{2}d_1 y_{101}y_{110}^2y_{1k-11}N(d_1)N(d_3)+$
	$\frac{1}{2}d_1 y_{101}^2y_{111}y_{1k-10}N(d_1)N(d_3)+$
	$\frac{1}{6}d_3 y_{101}^3y_{0k1}N(d_1)N(d_3)+$
	$\frac{1}{2}d_3 y_{101}^2y_{011}y_{1k-11})N(d_1)N(d_3)+$
	$\frac{1}{2}d_3 y_{101}^2y_{111}y_{0k-11}N(d_1)N(d_3)+$
	$\frac{1}{2}d_1\sum_{i,j\in\{2,\dots k-2\}} y_{101}^2y_{1i0}y_{1j0}g_{ji}N(d_1)N(d_3)+$
	$\frac{1}{2}d_3\sum_{i,j\in\{2,\dots k-2\}} y_{101}^2y_{0i1}y_{1j1}g_{ji}N(d_1)N(d_3)$
$(0,d_2,d_3)$	$\frac{1}{6}d_2 y_{011}^3y_{k10}N(d_2)N(d_3)+$
	$\frac{1}{2}d_2 y_{011}y_{110}^2y_{1k-11}N(d_2)N(d_3)+$
	$\frac{1}{2}d_2 y_{011}^2y_{111}y_{k-110}N(d_2)N(d_3)+$
	$\frac{1}{6}d_3 y_{011}^3y_{k01}N(d_2)N(d_3)+$
	$\frac{1}{2}d_3 y_{011}^2y_{101}y_{k-111})N(d_2)N(d_3)+$
	$\frac{1}{2}d_3 y_{011}^2y_{111}y_{k-101}N(d_2)N(d_3)+$
	$\frac{1}{2}d_2\sum_{i,j\in\{2,\dots k-2\}} y_{011}^2y_{i10}y_{j10}g_{ji}N(d_2)N(d_3)+$
	$\frac{1}{2}d_3\sum_{i,j\in\{2,\dots k-2\}} y_{011}^2y_{i01}y_{j11}g_{ji}N(d_2)N(d_3)$
(d_1,d_2,d_3)	$\frac{1}{2}d_1 y_{101}y_{110}y_{111}^2N(d_1)N(d_2)N(d_3)+$
	$\frac{1}{2}d_2 y_{011}y_{110}y_{111}^2N(d_1)N(d_2)N(d_3)+$
	$\frac{1}{2}d_3 y_{101}y_{010}y_{111}^2N(d_1)N(d_2)N(d_3)$

Summing up all $n = 5$ contributions:

β	contribution to Φ_β
$(d_1, d_2, 0)$	$\frac{1}{6}d_1^2 y_{101} y_{10k-1} y_{110}^3 N(d_1)N(d_2)+$
	$\sum_{i,j\in\{2,\dots k-2\}} \frac{1}{6}d_1^2 y_{10i} y_{10j} y_{110}^3 g_{ji} N(d_1)N(d_2)+$
	$\frac{1}{6}d_2^2 y_{011} y_{01k-1} y_{101} y_{01k-1} y_{110}^3 N(d_1)N(d_2)+$
	$\sum_{i,j\in\{2,\dots k-2\}} \frac{1}{6}d_2^2 y_{01i} y_{01j} y_{110}^3 g_{ji} N(d_1)N(d_2)+$
	$\frac{1}{6}d_1 d_2 y_{101} y_{01k-1} y_{110}^3 N(d_1)N(d_2)+$
	$\frac{1}{6}d_1 d_2 y_{011} y_{10k-1} y_{110}^3 N(d_1)N(d_2)+$
	$\sum_{i,j\in\{2,\dots k-2\}} \frac{1}{6}d_1 d_2 y_{10i} y_{01j} y_{110}^3 g_{ji} N(d_1)N(d_2)$
$(d_1, 0, d_3)$	$\frac{1}{6}d_1^2 y_{101}^3 y_{110} y_{1k-10} N(d_1)N(d_3)+$
	$\sum_{i,j\in\{2,\dots k-2\}} \frac{1}{6}d_1^2 y_{101}^3 y_{1i0} y_{1j0} g_{ji} N(d_1)N(d_3)+$
	$\frac{1}{6}d_3^2 y_{101}^3 y_{011} y_{0k-11} N(d_1)N(d_3)+$
	$\sum_{i,j\in\{2,\dots k-2\}} \frac{1}{6}d_3^2 y_{101}^3 y_{1i1} y_{1j1} g_{ji} N(d_1)N(d_3)+$
	$\frac{1}{6}d_1 d_3 y_{101}^3 y_{110} y_{0k-11} N(d_1)N(d_3)+$
	$\frac{1}{6}d_1 d_3 y_{101}^3 y_{1k-10} y_{011} N(d_1)N(d_3)+$
	$\sum_{i,j\in\{2,\dots k-2\}} \frac{1}{6}d_1 d_3 y_{101}^3 y_{1i1} y_{1j1} g_{ji} N(d_1)N(d_3)$
$(0, d_2, d_3)$	$\frac{1}{6}d_2^2 y_{011}^3 y_{110} y_{k-110} N(d_2)N(d_3)+$
	$\sum_{i,j\in\{2,\dots k-2\}} \frac{1}{6}d_2^2 y_{011}^3 y_{i10} y_{j10} g_{ji} N(d_2)N(d_3)+$
	$\frac{1}{6}d_3^2 y_{101}^3 y_{011} y_{0k-11} N(d_2)N(d_3)+$
	$\sum_{i,j\in\{2,\dots k-2\}} \frac{1}{6}d_3^2 y_{011}^3 y_{i11} y_{j11} g_{ji} N(d_2)N(d_3)+$
	$\frac{1}{6}d_2 d_3 y_{011}^3 y_{k-110} y_{101} N(d_2)N(d_3)+$
	$\frac{1}{6}d_2 d_3 y_{011}^3 y_{110} y_{k-101} N(d_2)N(d_3)+$
	$\sum_{i,j\in\{2,\dots k-2\}} \frac{1}{6}d_2 d_3 y_{011}^3 y_{i11} y_{j11} g_{ji} N(d_2)N(d_3)$
(d_1, d_2, d_3)	$\frac{1}{4}d_1^2 y_{101}^2 y_{110}^2 y_{111} N(d_1)N(d_2)N(d_3)+$
	$\frac{1}{4}d_2^2 y_{010}^2 y_{110}^2 y_{111} N(d_1)N(d_2)N(d_3)+$
	$\frac{1}{4}d_3^2 y_{101}^2 y_{011}^2 y_{111} N(d_1)N(d_2)N(d_3)+$
	$\frac{1}{2}d_1 d_2 y_{101} y_{010} y_{110}^2 y_{111} N(d_1)N(d_2)N(d_3)+$
	$\frac{1}{2}d_1 d_3 y_{101} y_{101} y_{110} y_{111} y_{011} N(d_1)N(d_2)N(d_3)+$
	$\frac{1}{2}d_2 d_3 y_{011}) y_{101} y_{110} y_{111} y_{011} N(d_1)N(d_2)N(d_3)+$

Finally the contribution to Φ_β for $\beta = (d_1, d_2, d_3)$ and $n = 6$ is:

$$
\begin{aligned}
&(\tfrac{1}{36}(d_1^3 y_{101}^3 y_{110}^3 + d_2^3 y_{011}^3 y_{110}^3 y_{110}^3) y_{101}^3 y_{011}^3 d_3^3 + \\
&\tfrac{1}{12}(d_1^2 d_2 y_{101}^2 y_{011} y_{110}^3 + d_1^2 d_3 y_{101}^3 y_{011} y_{110}^2 + d_1 d_3^2 y_{101} y_{101}^2 y_{011}^2 y_{110} + \\
&d_1 d_2^2 y_{101} y_{011}^2 y_{110}^3 + d_2^2 d_3 y_{101} y_{011}^3 y_{110}^2 + d_2 d_3^2 y_{011}) y_{101}^2 y_{011}^2 y_{110}) + \\
&\tfrac{1}{4} d_1 d_2 d_3 y_{101}^2 y_{011}^2 y_{110}^2) N(d_1)N(d_2)N(d_3)
\end{aligned}
$$

3.3.6. Remark. Since the final formula for the potential is rather long, we refrain from writing it down explicitly. It can, however, be easily produced from the presented tables.

We would like to emphasize that the above example shows that the product of two manifolds, whose quantum cohomology is already determined by the small quantum cohomology ring, may have a quantum cohomology ring which cannot be read off from the small one.

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