LAPLACE-BELTRAMI EIGENFUNCTIONS FOR DEFORMATION INVARIANT SHAPE REPRESENTATION

Raif Rustamov

Department of Mathematics

Purdue University, West Lafayette, IN

Motivation

- Deformable shapes
  - Computer graphics
  - Shape modeling
  - Medical imaging
  - 3D face recognition
- Achieve deformation/pose invariant
  - Retrieval/matching
  - Correspondence
  - Segmentation

General approach

- Natural articulations
  - pair-wise geodesic distances change little
  - isometries – metric tensor stays same
- Deformation invariant embedding
  - Only metric properties are used
  - Produce an embedding of the surface into (higher dimensional) Euclidean space
  - The object and its deformations have the same embedding
- Segmentation, descriptor extraction, etc. uses this embedding – deformation invariance is achieved

Geodesics based embeddings

- Spectral embedding – MDS, Jain-Zhang
  - Pairwise geodesic distances between points
  - Flatten this structure – get embedding
  - Euclidean distance in embedding = geodesic dist.
- Successful:
  - classification, correspondence, segmentation
- Problems:
  - Geodesic distances are sensitive to local topology changes
  - A “short circuit” can affect a lot of geodesics

Our approach

- Construct an embedding
  - Geodesic distances are never used
  - Laplace-Beltrami eigenfunctions guide the construction
- Eigenfunctions have global nature
  - more stability to local changes
- Eigenfunctions are isometry invariant
  - Deformation invariant representation

Laplace-Beltrami

- Egenvalues, eigenfunctions solve $\Delta\phi = \lambda\phi$
- Eigenvalues: $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_i < \ldots$
- Eigenfunctions: ϕ_i
  - Constitute an orthogonal basis
  - Bruno Levy: this basis is **the one**!

Global Point Signatures

- Given a point **p** on the surface we define

$$GPS(\mathbf{p}) = \left(\frac{1}{\sqrt{\lambda_1}}\phi_1(\mathbf{p}), \frac{1}{\sqrt{\lambda_2}}\phi_2(\mathbf{p}), \frac{1}{\sqrt{\lambda_3}}\phi_3(\mathbf{p}), \ldots \right)$$

- $\phi_i(\mathbf{p})$ is the value of the eigenfunction at the point **p**
- Reason for square roots will be explained later

GPS embedding

- GPS can be considered as a mapping from the surface into infinite dimensional space.
- The image of this map will be called the **GPS embedding** of the surface.
- The infinite dimensional ambient space the **GPS domain**

Property 1: distinctness

- A surface without self-intersections is mapped into a surface without self-intersections
- In other words: distinct points have distinct images under the GPS .

Property 2: invariance

- GPS embedding is an isometry invariant.
  - Two isometric surfaces will have the same image under the GPS mapping
  - Same GPS embedding
- Reason:
  - Laplace-Beltrami operator is defined completely in terms of the metric tensor
  - LB is isometry invariant
  - LB eigenvalues and eigenfunctions of isometric surfaces coincide - their GPS embeddings also coincide

Property 3: reconstruction

- Given the GPS embedding and the eigenvalues, one can recover the surface *up to isometry*
- Eigenvalues and eigenvectors of LB uniquely determine the metric tensor.
  - This stems from completeness of eigenfunctions, which implies the knowledge of Laplace-Beltrami, from which one immediately recovers the metric tensor and so, the isometry class of the surface.

Property 4

- GPS embedding is absolute: it is not subject to rotations or translations of the ambient infinite-dimensional space.
- Compare with Geodesic MDS embedding
  - Determined only up to translations and rotations
  - there is no uniquely determined positional normalization relative to the embedding domain.
  - In order to compare two shapes, one still needs to find the appropriate rotations and translations to align the MDS embeddings of the shapes

Property 4, cntd.

- The GPS embedding is uniquely determined
  - two isometric surfaces will have exactly the same GPS embedding
  - except for reflections, because the signs of eigenfunctions are not fixed
  - no rotation or translation in the ambient infinite dimensional space will be involved
  - Example: the center of mass of the GPS embedding will automatically coincide with the origin

Property 5: meaningful distance

- The inner product and, thereby, the Euclidean distance in the GPS domain have a meaningful interpretation
- Green's function G(**x**, **x'**)

$$G(\mathbf{x},\mathbf{x}') = \sum_{i=1}^{\infty} \frac{\phi_i(\mathbf{x})\phi_i(\mathbf{x}')}{\lambda_i}.$$

- The dot product in ambient space has meaning:

$$G(\mathbf{p},\mathbf{q}) = GPS(\mathbf{p}) \cdot GPS(\mathbf{q})$$

Discrete Setting

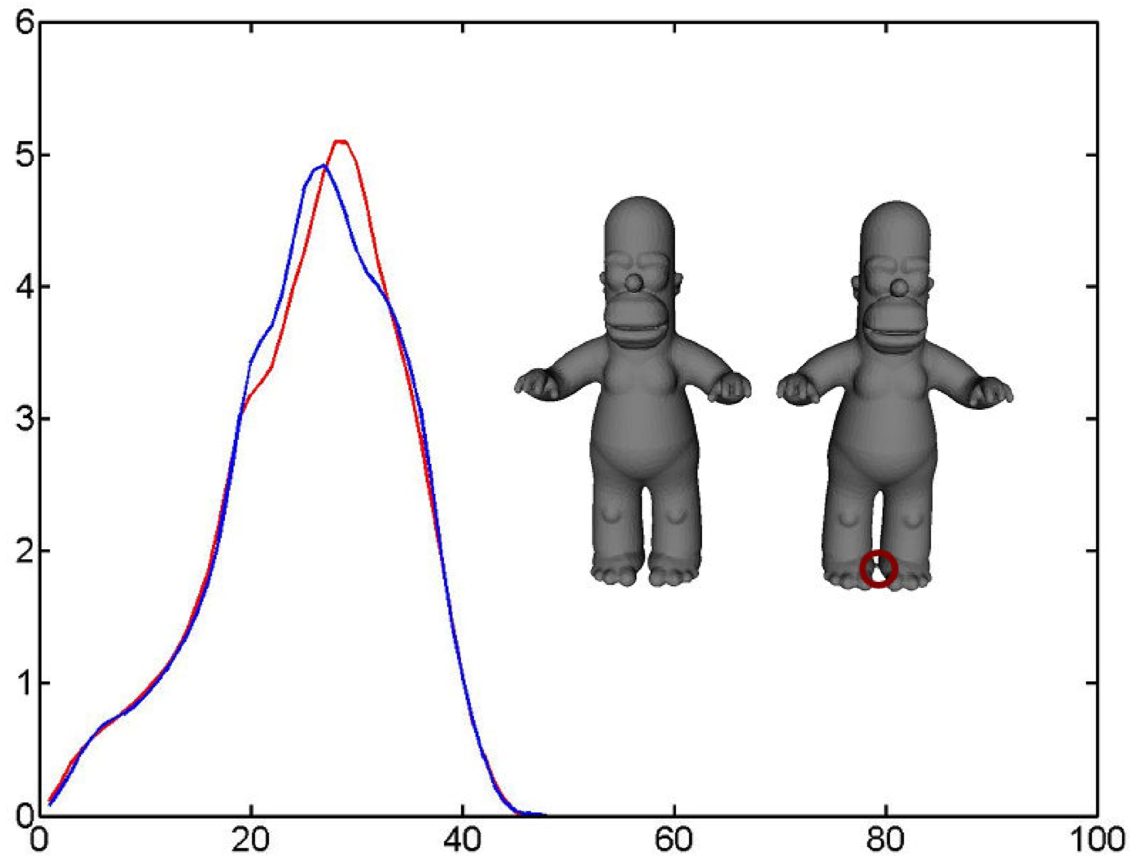
- Use Laplacian of Xu
- It is not symmetric
- We explain how to handle the non-symmetry
- Several novel remarks: complementary to "No Free Lunch":
  - Wardetzky et al. prove that there is no discrete Laplacian that satisfies a set of requirements **including symmetry**
  - We show that one should not require a Laplacian to be symmetric
  - Also see "Symmetric Laplacian Considered Harmful"

Experiments

- Deformable shape classification
- G2 distributions
  - A variant of D2, but computed on the GPS embedding
  - Automatically deformation (isometry) invariant

Stability

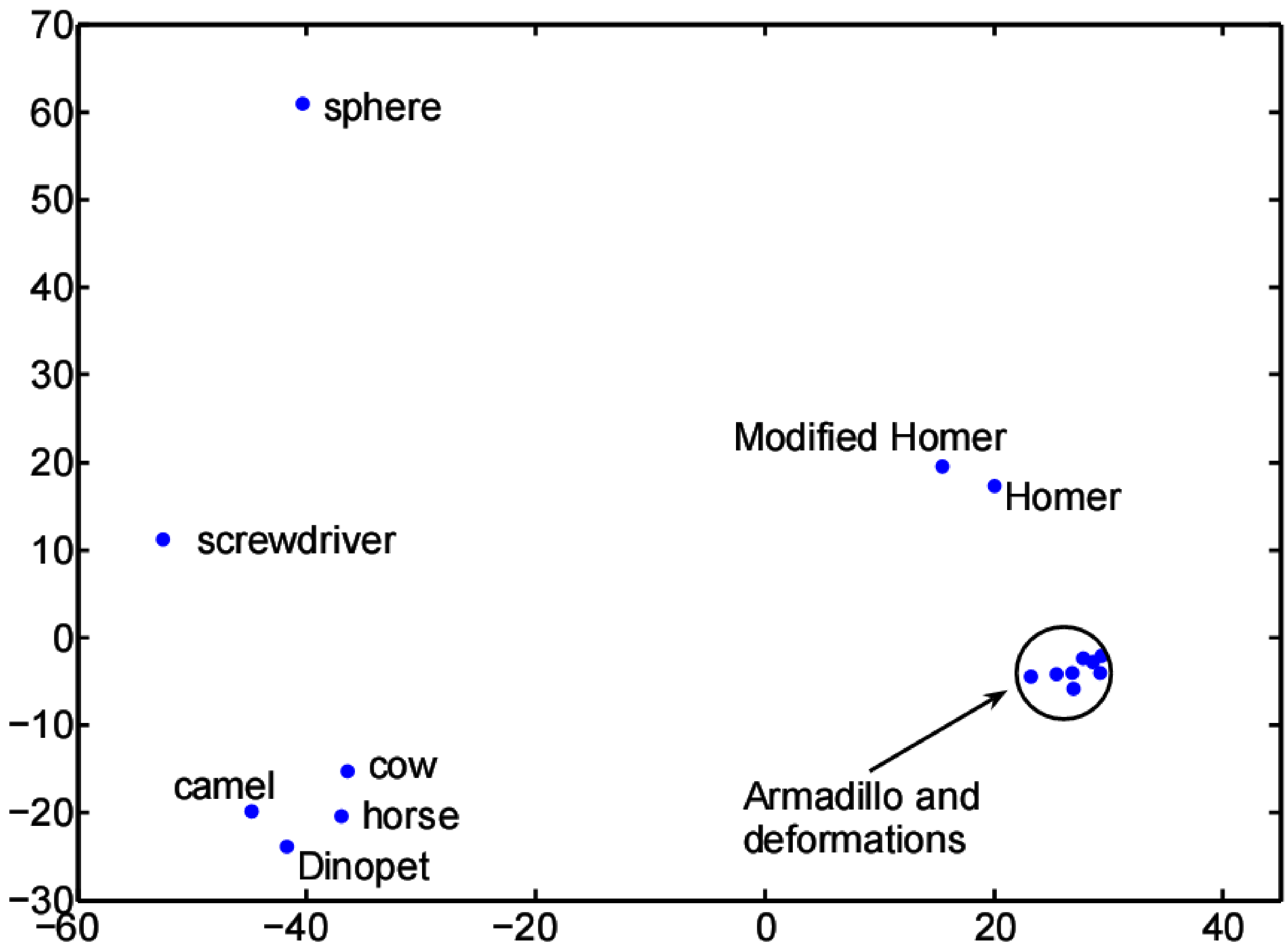
- The global nature of eigenfunctions makes the G2 stable under local topology changes: welded blue



Isometry invariance: dataset

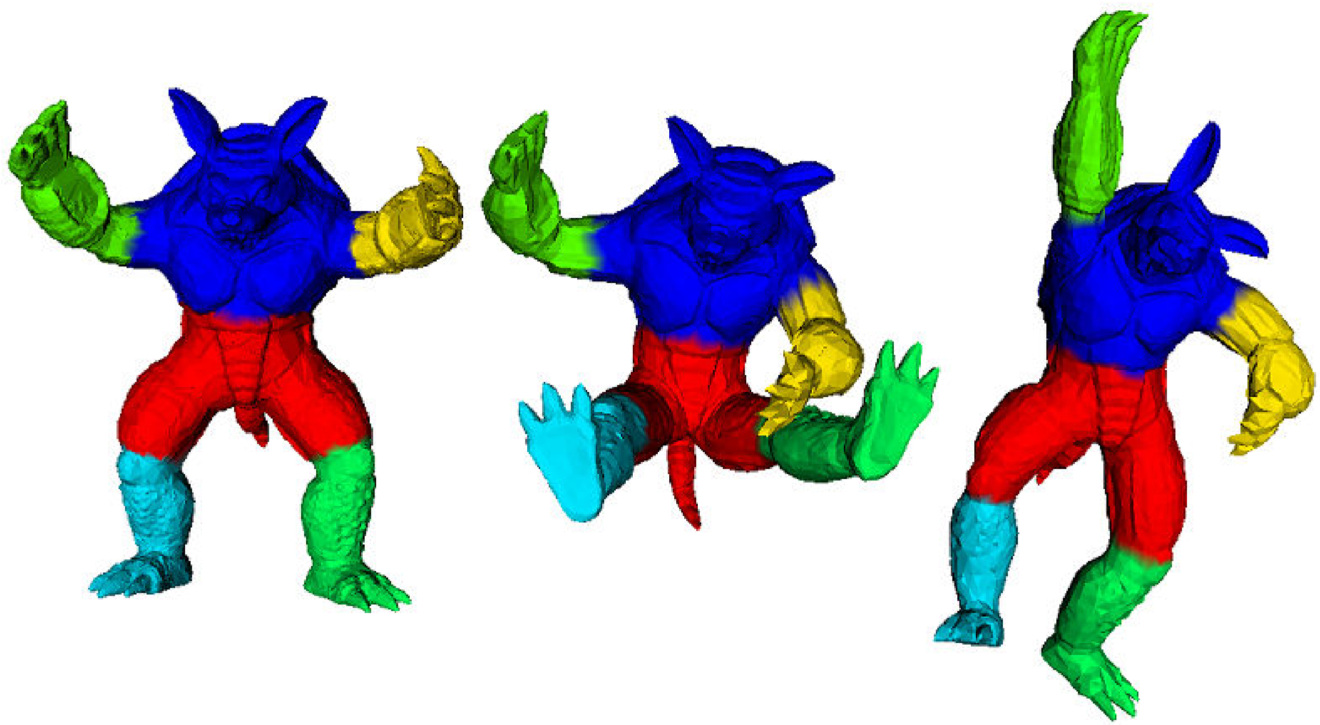
- Yoshizawa et al.

Isometry invariance: MDS plot



Sample segmentation

- K-means clustering in the GPS, not optimized



Problems

- Inability to deal with degenerate meshes
- Surfaces with boundaries
  - impose appropriate boundary conditions.
- Two problems while working with eigenvalues and eigenvectors in general:
  - the signs of eigenvectors are undefined
  - two eigenvectors may be swapped
- Using D2 distributions indirectly addresses both of these issues.
- Further analysis is needed to clarify the consequences of these factors for shape processing when the GPS embedding is used directly

Acknowledgements

- Doctor Steve Novotny for not putting my fractured finger into a cast – this paper would not be possible
- Anonymous reviewers for their detailed and useful comments -- helped improve the paper immensely
- All models except the Dinopet and the sphere are from AIM@SHAPE Shape Repository; Deformations of Armadillo courtesy Shin Yoshizawa; the rest of the models are courtesy of INRIA

- Thank You!