Identification of viscosity in an incompressible fluid

Xiaosheng Li

Department of Mathematics
University of Washington
xli@math.washington.edu

In this work we consider the unique determination of the viscosity in an incompressible fluid. Assume that a bounded domain is filled with an incompressible fluid and the velocity vector field satisfies the stationary Stokes system. We prove that the viscosity can be determined from the Cauchy data on the boundary of the domain. This is a joint work with H. Heck and J-N Wang.

We state this work more precisely. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Assume that Ω is filled with an incompressible fluid. Let $u = (u^1, u^2, u^3)^T$ be the velocity vector field satisfying the stationary Stokes system

$$\begin{cases} \text{div}\sigma_\mu(u,p) = 0 \quad \text{in} \quad \Omega \\ \text{div}u = 0 \quad \text{in} \quad \Omega \end{cases} \tag{1}$$

where $\sigma_\mu(u,p) = 2\mu\text{Sym}(\nabla u) - pI$ and $\text{Sym}(A) = (A + A^T)/2$ is the symmetric part of the matrix A. Here $\mu(x) > 0$ is the viscosity function. Now let $\phi \in H^{1/2}(\partial\Omega)$ satisfy the compatibility condition

$$\int_{\partial\Omega} \phi \cdot \mathbf{n} ds = 0$$

with $\mathbf{n}$ being the unit outer normal of $\partial\Omega$, then there exists a unique $(u,p) \in H^1(\Omega) \times L^2(\Omega)$ (p is unique up to a constant) solving (1) and $u|_{\partial\Omega} = \phi$. So we can define the Cauchy data of (u,p) satisfying (1)

$$S_\mu = \{(u|_{\partial\Omega}, \sigma_\mu(u,p)\mathbf{n}|_{\partial\Omega})\} \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega).$$

In the physical sense, $\sigma_\mu(u,p)\mathbf{n}|_{\partial\Omega}$ represents the Cauchy force acting on the boundary of the domain. The inverse problem is to determine μ from the knowledge of S_μ. We prove the following global uniqueness result.

Theorem 1. *Let $\partial\Omega$ be convex with nonvanishing Gauss curvature. Assume that $\mu_1(x)$ and $\mu_2(x)$ are two viscosity functions satisfying $\mu_1, \mu_2 \in C^{n_0}(\overline{\Omega})$ for $n_0 \geq 8$. Let S_{μ_1} and S_{μ_2} be the Cauchy data associated with μ_1 and μ_2, respectively. If $S_{\mu_1} = S_{\mu_2}$ then $\mu_1 = \mu_2$.*