MA 544 12/09/09 FINAL EXAM KRAN G018 WED 12/16 8:00 A.M. 1

Here $(X, \mathcal{S})$ will denote a measurable space.

Def. If μ and ν are signed measures on $\mathcal{S}$, we say that ~~μ~~ ν is singular with respect to μ if one can write

$$X = A \cup B, \quad A, B \in \mathcal{S}$$

$$A \cap B = \phi \; ; \quad |\mu|(A) = 0 \quad \text{and} \quad |\nu|(B) = 0$$

Theorem: Let μ, ν be σ-finite signed measures on $(X, \mathcal{S})$. Then there exist σ-finite signed measures ν_0 and ν_1 such that

$$\nu = \nu_0 + \nu_1, \quad \nu_0 \perp \mu \quad \text{and} \quad \nu_0 << \mu.$$

Moreover this decomposition is unique.

Proof: We may replace μ by $|\mu|$ and assume that μ is a measure. We may also write

$$X = \bigcup_{j=1}^{\infty} X_j \quad X_j \cap X_k = \phi, \quad |\mu|(X_j) < \infty$$
$$|\nu|(X_j) < \infty$$

Finally we may work with ν^+ and ν^- separately and hence assume that ν is a measure. So we have reduced the ~~proof~~ proof of the result to the case when

μ and ν are (positive) measures and

$\mu(X) < \infty$, $\nu(X) < \infty$.

In this case $\nu << \mu + \nu$ and thus, by Radon-Nikodym theorem. there exists

$f : X \longrightarrow [0, \infty)$ such that

$$\nu(E) = \int_E f \, d\mu + \int_E f \, d\nu \leq \mu(E) + \nu(E)$$

Hence $$\int_E f \, d(\mu+\nu) \leq \int_E 1 \, d(\mu+\nu)$$

$\forall E \in \mathcal{T} \Rightarrow f \leq 1$ a.e w.r.t $\mu+\nu$.

Since $\nu << \mu+\nu$, $f \leq 1$ a.e. w.r.t ν.

Let $A = \{x : f(x) = 1\}$ and $B = X \setminus A$

Hence

$$\gamma(A) = \int_A f\,d\mu + \int_A f\,d\gamma = \gamma(A) + \mu(A).$$

Since $\gamma(A) < \infty$, $\mu(A) = 0$.

Now let $\gamma_0(E) = \gamma(E \cap A)$; $\gamma_1(E) = \gamma(E \cap B)$.

Hence $\gamma = \gamma_0 + \gamma_1$ and $\gamma_0 \perp \mu$. Now

We need to prove that $\gamma_1 << \mu$.

Let $E \in \mathcal{J}$ be such that $\mu(E) = 0$

then

$$\gamma_1(E) = \gamma(E \cap B) = \int_{E \cap B} f\,d\mu + \int_{E \cap B} f\,d\gamma.$$

Since $\mu(E) = 0$ $\int_{E \cap B} f\,d\mu = 0$, then

$$\int_{E \cap B} (1-f)\,d\gamma = 0$$

But on $E \cap B$; $f < 1$ a.e. therefore.

$$\gamma(E \cap B) = 0 \implies \gamma_1(E) = 0$$

To prove uniqueness we notice that if

$\gamma = \gamma_0^1 + \gamma_1^1 = \gamma_0^2 + \gamma_1^2$ $\quad$ $\gamma_0^j \perp \mu$ and

$\gamma_1^j << \mu$.

$$\gamma_0^1 - \gamma_0^2 = \gamma_1^2 - \gamma_1^1$$

But $(\gamma_0^1 - \gamma_0^2) \perp \mu$ and $\gamma_1^2 - \gamma_1^1 << \mu$.

So $\gamma_0 = \gamma_0^2$ and $\gamma_1^2 - \gamma_1^1 = 0$

Note: If $\gamma_1 \perp \mu$ and $\gamma_2 \perp \mu$, then $\gamma_1 + \gamma_2 \perp \mu$.

$X = A_1 \cup B_1$; $|\mu|(A_1) = 0$; $|\gamma_1|(B_1) = 0$

$X = A_2 \cup B_2$; $|\mu|(A_2) = 0$; $|\gamma_2|(B_2) = 0$

$$X = (A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (B_1 \cap A_2) \cup (B_1 \cap B_2)$$

$$|\mu|\left((A_1 \cap A_2) \cup (A_1 \cap B_2)\right) = 0$$

$$|(\gamma_1 + \gamma_2)|\left((B_1 \cap A_2) \cup (B_1 \cap B_2)\right) = 0.$$

Linear Functionals on $L^p(X)$

$(X, \mathcal{S}, \mu)$ a measure space. $1 \le p < \infty$

$$L^p(X) = \{ [f] : \left(\int |f|^p d\mu \right)^{1/p} = \|f\|_p < \infty \}$$

Hölder's Inequality if $f \in L^p(X)$; $g \in L^q(X)$
$1/p + 1/q = 1$

$$\left| \int f g \, d\mu \right| \le \int |f g| \, d\mu \le \|f\|_p \cdot \|g\|_q .$$

Therefore for each $g \in L^q(X)$, we define

a map

$$x_g : L^p(X) \longrightarrow \mathbb{C}$$
$$f \longmapsto \int f g \, d\mu .$$

This is called a linear functional.

Notice that

$$| x_g(f) | \le \|f\|_p \cdot \|g\|_q .$$

We say that a linear functional $\ell : L^p(X) \longrightarrow \mathbb{C}$ is bounded if there exists a constant C such that

$$|\ell(f)| \leq C\|f\|_p \quad \forall\, f \in L^p(X)$$

We define

$$\|\ell\| = \sup_{\|f\|_p = 1} \frac{|\ell(f)|}{\|f\|_p}$$

<u>Proposition</u>: If $g \in L^q(\mathbb{R}^n)$, then

$$\|\chi_g\| = \|g\|_{L^q}.$$

<u>Proof</u>: We want to show that

$$\sup_{\|f\|_p = 1} \left| \int f g \right| d\mu = \|g\|_{L^q}.$$

We know that for every $f \in L^p(X)$,

$$\left| \int f g \, d\mu \right| \leq \|g\|_q \cdot \|f\|_p.$$

and therefore if $\|f\|_p = 1$.

$$\sup_{\|f\|_p=1} |x_g(f)| \leq \|g\|_q .$$

Of course we may assume $\|g\|_q > 0$, otherwise this is obvious. In this case let

$$f(x) = \begin{cases} 0 & \text{if } g(x) = 0 \\ \dfrac{\bar{g}}{|g|} \cdot \dfrac{|g|^{q-1}}{\|g\|_q^{q/p}} \end{cases}$$

$$\int g f \, d\mu = \int \frac{|g|^q}{\|g\|_q^{q/p}} \, d\mu = \frac{1}{\|g\|_q^{q/p}} \cdot \|g\|_q^q$$

$$= \|g\|_q^{q(1-1/p)} = \|g\|_q .$$

$$\|f\|_p^p = \frac{1}{\|g\|_q^q} \int |g|^{(q-1)p} \, d\mu = \frac{1}{\|g\|_q^q} \cdot \int |g|^q d\mu = 1 .$$

Theorem: Let $(X, \mathcal{S}, \mu)$ be a σ-finite space and $1 < p < \infty$. Let q be the conjugate to p. Then to every bounded linear functional $l : L^p(X) \longrightarrow \mathbb{C}$, there exists a unique $g \in L^q(X)$ such that

$$l(f) = \int f g \, d\mu \quad \forall f \in L^p(X).$$

Moreover $\|l\| = \|g\|_q$.