

EXAM I

3

2)(20 points) Show that if $f \in L^1(X, \mu)$, $f \geq 0$, and $g : X \rightarrow [0, \infty]$ is measurable and $q \geq 1$, then

$$\left(\int_X g f d\mu \right)^q \leq \|f\|_{L^1}^{q-1} \int_X g^q f d\mu.$$

Solution. Write. (if $\|f\|_1 > 0$)

$$\int g f d\mu = \|f\|_1 \int g \frac{f}{\|f\|_1} d\mu.$$

Then Jensen's inequality implies

$$\text{for } \frac{f}{\|f\|_1} d\mu$$

$$\left(\int_X g f d\mu \right)^p = \left(\|f\|_1 \int g \frac{f}{\|f\|_1} d\mu \right)^p \leq$$

$$\leq \|f\|_1^p \int g^p \frac{f}{\|f\|_1} d\mu = \|f\|_1^{p-1} \int_X g^p f d\mu.$$

EXAM I

4

3)(20 points) Let $f : X \rightarrow [0, \infty]$ be measurable and let ν is the measure defined in Problem 1. Let $g : X \rightarrow [0, \infty]$ and $p \in (1, \infty)$ and suppose that $\mu(\{g > t\}) \leq Ct^{-k}\nu(\{g > t\})$, $1 \leq k \leq p-1$. Show that

$$\|g\|_{L^p(\mu)}^k \leq \frac{Cp}{p-k} \|f\|_{L^{p/k}(\mu)},$$

provided the norms are finite.

Hint: Recall a homework problem: If (Z, \mathcal{M}, η) is a measure space and $f : Z \rightarrow [0, \infty]$ is measurable, then

$$\int_Z f^p d\eta = \int_0^\infty p t^{p-1} \lambda_f(t) dt,$$

where $\lambda_f(t) = \eta(\{x : f(x) > t\})$, $p \geq 1$, and $\int_Z f^p d\eta < \infty$.

Solution:

$$\int g^p d\mu = p \int_0^\infty t^{p-1} \mu(\{g > t\}) dt \leq Cp \int_0^\infty t^{p-1-k} \nu(\{g > t\}) dt$$

$$= \frac{Cp}{p-k} \int_0^\infty (p-k) t^{p-k-1} \nu(\{g > t\}) dt =$$

$$= \frac{Cp}{p-k} \int g^{p-k} f d\mu \quad \text{Using Hölder's inequality and that } \frac{k}{p} + \frac{p-k}{p} = 1$$

$$\leq \frac{Cp}{p-k} \cdot \left(\int g^p d\mu \right)^{1-\frac{k}{p}} \cdot \left(\int f^{p/k} d\mu \right)^{k/p}$$

$$\Rightarrow \left(\int g^p d\mu \right)^{k/p} \leq \frac{Cp}{p-k} \|f\|_{L^{p/k}}^{k/p}$$

EXAM I

5

4)(30 points) Let $f_n : X \rightarrow [0, \infty]$ $n \in \mathbb{N}$, and suppose that every subsequence of f_n has a subsequence which converges to a function f a.e. Moreover, suppose that there exist $g_n \in L^1(X)$ such that $f_n \leq g_n$, $n \in \mathbb{N}$ and that g_n converges in $L^1(X)$. Prove that

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

Solution. First pick a subsequence f_{n_j} of f_n such that $\lim_{n_j \rightarrow \infty} \int f_{n_j} \, d\mu = \liminf_{n \rightarrow \infty} \int f_n \, d\mu$. Then pick a subsequence $f_{n_{j_k}}$ of f_{n_j} such that $f_{n_{j_k}} \rightarrow f$ a.e. Then Fatou's lemma applied to $f_{n_{j_k}}$ gives.

$$(*) \quad \int_X f \, d\mu \leq \liminf_{n \rightarrow \infty} \int f_n \, d\mu.$$

Next let $F_n = g_n - f_n \geq 0$. ~~Repeat~~ Repeat the argument used above and construct a subsequence F_{n_j} such that $F_{n_j} \rightarrow g - f$ a.e. and $\lim_{n_j \rightarrow \infty} \int F_{n_j} \, d\mu = \liminf_{n \rightarrow \infty} \int F_n \, d\mu$.

Again, Fatou's lemma applied to F_{n_j} gives.

$$\int_X (g - f) \, d\mu \leq \liminf_{n \rightarrow \infty} \int (g_n - f_n) \, d\mu = \int g \, d\mu - \limsup_{n \rightarrow \infty} \int f_n \, d\mu$$

$$x \quad \text{Hence} \quad \int f \, d\mu \geq \limsup_{n \rightarrow \infty} \int f_n \, d\mu \quad (**)$$

(*) and (**) give that $\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu$.