MA 544- FIRST HOMEWORK – FALL 2009
DUE 9/4/2009

STUDENT NAME:

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1) (10) Let $F \subset \mathbb{R} = \{x_j, \ \ j \in \mathbb{N}\} \subset \mathbb{R}$ be closed. Show that F must have an isolated point.
2) Let $\mathbb{Q} = \{x_n, \ \ n = 1, 2, ...\}$ be an enumeration of the rational numbers.
a)(10) Show that the function

$$f(x) = \sum_{\{n: x_n < x\}} \frac{1}{2^n}$$

is strictly increasing and that $D(f) = \mathbb{Q}$.
3) (10) Let $f : [a, b] \longrightarrow [0, \infty)$ be bounded and $f \in \mathcal{R}$. Show that

$$\int_a^b f \, dx = \sup_g \int_a^b g \, dx, \ \ g \text{ is continuous and } \ \ g(x) \leq f(x), \ \ \forall \ \ x \in [a.b].$$

4) (10) Let $f : [a, b] \longrightarrow [0, \infty)$ be bounded. Let $\omega(f, x)$ be the oscillation of f at the point x. Show that

$$\overline{\int_a^b} f(x) \, dx - \underline{\int_a^b} f(x) \, dx = \overline{\int_a^b} \omega(x) \, dx$$

5)(10) Let $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. Let $t_j \in [x_{j-1}, x_j]$ and form the Riemann sum

$$\Sigma(f, P) = \sum_{j=1}^{n} f(t_j)(x_1 - x_{j-1}).$$

Let $|P| = \max\{x_j - x_{j-1}, \ \ 1 = 1, 2, ...n\}$. Prove that $\lim_{|P| \to 0} \Sigma(f, P)$ exists if and only if $f \in \mathcal{R}$, and in that case

$$\lim_{|P| \to 0} \Sigma(f, P) = \int_a^b f \, dx.$$

6) Let $f : [a, b] \longrightarrow \mathbb{R}$ with $f \in \mathcal{R}$. For $x \in [a, b]$ Let

$$F(x) = \int_a^x f \, ds.$$

a)(10) Show that F may not be differentiable at every $x \in [a, b]$.
b)(10) If f is continuous at c show that F is differentiable at c and $F'(c) = f(c)$.
6)(10) Let $F : [a, b] \longrightarrow \mathbb{R}$ be differentiable and suppose that $F' \in \mathcal{R}$. Show that

$$F(b) - F(a) = \int_a^b F'(x) \, dx.$$

7) (10) State and prove Baire category theorem.