MATH 544 — FALL 2009 — SECOND HOMEWORK
DUE SEPTEMBER 21, 2009

STUDENT NAME

STUDENT NUMBER

1) (15 points) (MA 544 Qual Exam, August 2005) Let $(X, \mathcal{M})$ be a measurable space, let μ be a positive measure on $\mathcal{M}$ such that $\mu(X) < \infty$. Let $f : X \longrightarrow \mathbb{R}$ be measurable with $f > 0$ almost everywhere. Let E_n, $n \in \mathbb{N}$, be a sequence of measurable sets in X with the property that

$$\lim_{n\to\infty} \int_{E_n} f \, d\mu = 0.$$

Prove that $\lim_{n\to\infty} \mu(E_n) = 0$.

2)(15 points) (MA 544 Qual Exam, August 2005) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Fix $1 \leq n \leq m$ and let $E_1, ..., E_m$ be measurable sets with the property that almost every $x \in X$ belongs to and least n of these sets. Prove that at least one of these sets must have measure greater than or equal to n/m.

3) (10 points) Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\int_X |f| \, d\mu < \infty$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| \, d\mu < \epsilon$ whenever $\mu(E) < \delta$.

4) (15 points) Let $f_n : X \longrightarrow [0, \infty]$, $n = 1, 2, ...$ be a sequence of measurable functions. Show that the set of points at which the sequence converges is measurable.

5)(15 points) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $g_n : X \longrightarrow [0, \infty]$, $n \in \mathbb{N}$, be such that $\int_X g_n \, d\mu < \infty$ for every n. Suppose there exists $g : X \longrightarrow [0, \infty]$ such that

$$\lim_{n\to\infty} \int_X |g_n - g| \, d\mu = 0.$$

Let $f_n : X \longrightarrow [0, \infty]$ be another sequence measurable functions on $(X, \mathcal{M}, \mu)$. If $f_n \leq g_n$ a.e. for every n, prove that

$$\limsup_{n\to\infty} \int f_n \, d\mu \leq \int \limsup_{n\to\infty} f_n \, d\mu.$$

6)(15 points)(MA 544 Qual Exam, Jan 2009) A measure space $(X, \mathcal{M}, \mu)$ is said to be σ-finite if there exist $E_j \in \mathcal{M}$ such that $\mu(E_j) < \infty$ and $X = \bigcup_{n=1}^{\infty} E_j$. Let $f : X \longrightarrow \mathbb{R}$ be such that $\int_X |f| \, d\mu < \infty$ and that $f(x) \neq 0$ for almost every $x \in X$. Prove that $(X, \mathcal{M}, \mu)$ is σ-finite.

7) (15 points) Let $(X, \mathcal{S}, \mu)$ is a measure space and let $f : X \longrightarrow [0, \infty]$ be measurable. Let $\lambda_f(\alpha) = \mu(\{x : f(x) > \alpha\})$. Prove that if $p \geq 1$, and $\int_X f^p \, d\mu < \infty$,

$$\int_X f^p \, d\mu = \int_{[0,\infty)} p \, \alpha^{p-1} \lambda_f(\alpha) \, d\alpha,$$

where the integral on the right hand side is the Riemann integral.
Hint: Show that if $c > 0$ and Q is measurable, the result holds for $c\chi_Q$.