MATH 544 — FALL 2009 — HOMEWORK 3
Due October 5, 2009

STUDENT NAME

STUDENT NUMBER

1) (10 points) Let $\mathcal{B}(\mathbb{R}^n)$ denote the Borel sets on $\mathbb{R}^n$. Prove that the Lebesgue measure μ is the only (positive) measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ which is invariant under translations and $\mu(I) = \mathrm{vol}(I)$ for every interval I.

2)(10 points) (MA 544, Qual, Aug 2005) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be Lebesgue measurable and such that

$$\int_a^b f(x)\,dx \geq 0, \quad \text{for all } a, b \in \mathbb{R}, \quad a \leq b.$$

Show that $f(x) \geq 0$ almost everywhere.

3) a)(10 points) Let A be a Lebesgue measurable set with $\mu(A) < \infty$. Let

$$\phi(x) = \mu\left(A \cap (-\infty, x]\right).$$

Show that ϕ is continuous at each $x \in \mathbb{R}$.

b) (10 points) (MA 544 Qual, August, 2004) Let $A \subset \mathbb{R}$ be a Lebesgue measurable set and let $0 \leq b \leq m(A)$. Show that there exists a Lebesgue measurable set $B \subset A$ such that $m(B) = b$.

c) (10 points) (MA 544 Qual, January 2001) Let E be a Lebesgue measurable subset of $[0, 1]$, and let μ be the Lebesgue measure. Demonstrate the existence of a point $t \in (0, 1)$ such that

$$\mu\left(\{x : x \in E \text{ and } x < t\}\right) = 1/2\mu(E).$$

4) (10 points) (# 3.34, Torchinsky's book) Construct a Borel set $E \subset \mathbb{R}$ such that $0 < \mu(E \cap I) < m(I)$ for every non-empty interval $I \subset \mathbb{R}$.

5)(10 points) (# 3.37, Torchinsky's book) Prove that if A is a Lebesgue measurable set, then

$$\mu(A) = \sup\{\mu(K) : K \text{ is compact and } K \subset A\}.$$

6) (10 points) (# 3.1, Torchinsky's book) Suppose A and B are not Lebesgue measureable, is the same true of $A \cup B$?

7) (10 points) (MA544 Qual, August 2009) Let $A \subset \mathbb{R}$ be a Lebesgue measurable set and let m denotes the Lebesgue measure. Suppose that there exists $\delta > 0$ such that for each interval $I \subset \mathbb{R}$, $m(A \cap I) \geq \delta m(I)$. Show that $m(A^c) = 0$, where $A^c = \mathbb{R} \setminus A$.

8) (10 Points) (MA 544 Qual, August 2009) Let f be continuous on $\mathbb{R}$ and assume that $\int |f|\,dx < \infty$. Compute the limit

$$\lim_{n\to\infty} \int_a^\infty \frac{n^{\frac{1}{2}} f(t)}{1 + nt^2}\,dt \text{ when } a < 0, a = 0 \text{ and } a > 0.$$