MATH 544 — FALL 2009 — HOMEWORK 4

DUE OCTOBER 26, 2009

STUDENT NAME

STUDENT NUMBER

Let $(X, \mathcal{M}, \mu)$ be a measure space.

1) a) (10 points) If $f_n \geq 0$ a.e., $n \in \mathbb{N}$, and f_n converges to f in measure to f. Prove that

$$\int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

b)(10 points) Suppose that f_n is measurable, $|f_n| \leq g$, $n \in \mathbb{N}$ and that $g \in L^1(X, \mu)$. If $f_n \to f$ in measrure, show that

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

c)(10 points) If $f, f_n \in L^p(X, \mu)$, $1 \leq p < \infty$. Suppose $f_n \to f$ a.e and $||f_n||_p \to ||f||_p$. Prove that $||f - f_n||_p \to 0$. Is this still true if we replace convergence a.e with convergence in measure?

2)(10 points) If $\mu(X) < \infty$, show that $f_n \to f$ in measure if and only if

$$\lim_{n\to\infty} \int_X \frac{|f_n - f|}{1 + |f_n - f|} \, d\mu = 0.$$

3) If $\mu(X) < \infty$ prove that

i) (5 points) If $f_n \to f$ a.e, then $f_n \to f$ in measure.

ii) (5 points) If $f_n \in L^p(X, \mu)$, with $1 \leq p \leq \infty$, and $||f_n - f||_p \to 0$, then $f_n \to f$ in measure.

iii) (5 points) Are the converses of i and ii true?

iv) (5 points) Suppose $X = \mathbb{R}$ equipped with the Lebesgue measure. Are i and ii true?

4)(10 points) Suppose $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is such that

$$\phi\left(\int_0^1 f(x) \, dx\right) \leq \int_0^1 \phi(f) \, dx$$

for every real bounded measurable f. Prove that ϕ is convex.

5)(10 points) Suppose $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Let $T_h f(x) = f(x + h)$. Show that

$$\lim_{h\to 0} ||f - T_h f||_p = 0.$$

6) (10 points) Suppose $\mu(X) < \infty$, $f \in L^\infty(X, \mu)$, $||f||_\infty > 0$. Let

$$\alpha_n = \int_X |f|^n \, d\mu, \quad n = 1, 2, ...$$

Prove that $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = ||f||_\infty$.

7) Let $f : X \longrightarrow \mathbb{C}$ be measurable, $||f||_\infty > 0$, and let $\phi(p) = \int_X |f|^p \, d\mu$, and let

$$E = \{p \in (0, \infty) : \phi(p) < \infty\}.$$

a)(5 points) If $r, s \in E$ and $r, s \in E$, prove that $[r, s] \subset E$.

b)(5 points) Show that $\log \phi$ is convex in the interior of E and ϕ is continuous on E.

c) (5 points) If $r < p < s$ show that $||f||_p \leq \max(||f||_r, ||f||_s)$ and that $L^r(X) \cap L^s(X) \subset L^p(X)$.

d) (5 points) Suppose there exists r such that $||f||_r < \infty$. Prove that $\lim_{p \to \infty} ||f||_p = ||f||_\infty$.