1. The Lebesgue Measure

Now we will discuss the Lebesgue measure. The reference for this part is the book *Real Variables,* by A.Torchinsky, which is on reserve at the library.

A closed interval in $\mathbb{R}^n$ is a set of the form

$$I = [a_1, b_1] \times ... \times [a_n, b_n], \quad -\infty < a_j < b_j < \infty, \quad 1 \leq j \leq n.$$

Similarly an open interval is a set of the form

$$I = (a_1, b_1) \times ... \times (a_n, b_n), \quad -\infty < a_j < b_j < \infty, \quad 1 \leq j \leq n.$$

The volume of either the open or closed interval is

$$v(I) = \prod_{j=1}^{n} (b_j - a_j).$$

The following lemma is an important part in the construction of the Lebesgue measure.

Lemma 1.1. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a nonempty open subset. Then there exist closed intervals $\{I_j, \quad j \in \mathbb{N}\}$ such that $\overset{\circ}{I}_j \cap \overset{\circ}{I}_k = \emptyset$, $j \neq k$, and $\mathcal{O} = \bigcup_{j=1}^{\infty} I_j$.*

Proof. For $k \in \mathbb{N}$, let P_k be the set of points of $\mathbb{R}^n$ whose coordinates are integral multiples of 2^{-k}. For each $p = (p_1, p_2, ..., p_n) \in P_k$ let

$$I(p, 2^{-k}) = \{x \in \mathbb{R}^n : \ p_j \leq x_j < p_j + 2^{-k}, \ 1 \leq j \leq n\}.$$

Let $\Omega_k = \{I(p, 2^{-k}) : \quad p \in P_k\}$.

These sets satisfy the following properties:

1) Fixed k, then for each point $x \in \mathbb{R}^n$ there exists *a unique* interval $I(p, 2^{-k})$ in Ω_k such that $x \in I(p, 2^{-k})$.
2) Let $k' > k$. If $I(p_1, 2^{-k}) \in \Omega_k$, $I(p_2, 2^{-k'}) \in \Omega_{k'}$ then either $I(p_2, 2^{-k'}) \subset I(p_1, 2^{-k})$ or $I(p_2, 2^{-k'}) \cap I(p_1, 2^{-k}) = \emptyset$.

To deduce property 1, just notice that any $q \in \mathbb{R}$ can be written as $q = m2^{-k} + m_1$ where $m \in \mathbb{N}$ and $|m_1| < 2^{-k}$.

To prove property 2, notice that it follows from property 1 that there exists a unique box $I(q, 2^{-k}) \in \Omega_k$ such that the corner $p_2 \in I(q, 2^{-k})$. Then of course, $I(p_2, 2^{-k'}) \subset I(q, 2^{-k})$. So either $q = p_1$, and in that case $I(p_2, 2^{-k'}) \subset I(p_1, 2^{-k})$ or $q \neq p_1$ and $I(p_2, 2^{-k'}) \cap I(p_1, 2^{-k}) = \emptyset$.

Now we can prove the lemma. Since $\mathcal{O}$ is open, then for every $x \in \mathcal{O}$ there exists $I(p, 2^{-k})$ such that $x \in \overline{I}(p, 2^{-k}) \subset \mathcal{O}$. Thus if $\mathcal{F} = \{I(p, 2^{-k}) \in \Omega_k : \overline{I}(p, 2^{-k}) \subset \mathcal{O}\}$,

$$\mathcal{O} = \bigcup_{I(p,2^{-k}) \in \mathcal{F}} I(p, 2^{-k}).$$

Now we just filter the collection, i.e. take those intervals in Ω_1 and throw out all the intervals in Ω_j, $j > 1$, which are contained in one of the intervals in Ω_1 which are contained in $\mathcal{O}$. From the remaining collection select the intervals in Ω_2 which are contained in $\mathcal{O}$ and remove the ones in Ω_j, $j > 2$, which are contained in any of these. Repeat the process for $n = 3, 4, ...$ The remaining intervals $\{I_j, \ j \in \mathbb{N}\}$ are disjoint and satisfy $\mathcal{O} = \bigcup_{j=1}^{\infty} I_j$. To finish the proof of the lemma just take the closures of these intervals. $\square$

Given a set $A \subset \mathbb{R}^n$ we define its outer measure by

$$|A|_e = \inf\{\sum_k v(I_k) : A \subset \bigcup_{k=1}^{\infty} I_k\}.$$

Here the infimum is taken over the family of countable covers of A by closed intervals.

Theorem 1.1. *The outer measure has the following properties:*
P.1) If $A \subset B$, then $|A|_e \le |B|_e$.
P.2) If I_j, $j = 1, 2, ..., N$ be either closed intervals with $\overset{\circ}{I}_k \cap \overset{\circ}{I}_j = \emptyset$ if $k \neq j$, or pairwise disjoint open intervals, then

$$\left| \bigcup_{j=1}^{N} I_j \right|_e = \sum_{j=1}^{N} v(I_j).$$

P.3) Let $E_k \subset \mathbb{R}^n$, $k = 1, 2, ...$ Then

$$\left| \bigcup_{k=1}^{\infty} E_k \right|_e \le \sum_{k=1}^{\infty} |E_k|_e .$$

P.4) For any $E \subset \mathbb{R}^n$,

$$|E|_e = \inf\{|\mathcal{O}|_e : E \subset \mathcal{O} \ \text{ and } \ \mathcal{O} \text{ is open } \}.$$

P.5) For any $E \subset \mathbb{R}^n$ there exists a G_δ set H such that $E \subset H$ and $|E|_e = |H|_e$.

Proof. To prove property $P.1$ just notice that any family of closed intervals that cover B also cover A. So the inequality just follows from the definition of infimum.

We first prove $P.2$ for closed intervals, and we begin by observing that the intervals I_k, $1 \le k \le N$ form a cover of the set $A = \bigcup_{j=1}^{N} I_j$. Therefore, by definition

$$|A|_e \le \sum_{j=1}^{N} v(I_j).$$

Now we want to prove that $|A|_e \ge \sum_{j=1}^{N} v(I_j)$. Given $\epsilon > 0$, it follows from the definition of infimum that there exist closed intervals J_m, $m = 1, 2, ...$, covering A, such that

$$|A|_e \le \sum_{m=1}^{\infty} v(J_m) < |A|_e(1+\epsilon). \tag{1.1}$$

Let us denote $J_m = [a_1^m, b_1^m] \times .... \times [a_n^m, b_n^m]$. Then $v(J_m) = \prod_{i=1}^{n}(b_i^m - a_i^m)$. Let $\delta_m > 0$ and let J'_m be the open interval $J'_m = (a_1^m - \delta_m, b_1^m + \delta_m) \times .... \times (a_n^m - \delta_m, b_n^m + \delta_m)$. Then $v(J'_m) = \prod_{i=1}^{n}(b_i^m - a_i^m + 2\delta_m)$. Since $-\infty < a_i^m < b_i^m < \infty$, $i = 1, 2, ..., n$, one can pick δ_m such that

$$v(J'_m) < (1+\epsilon)v(J_m). \tag{1.2}$$

Since $A \subset \bigcup_{m=1}^{\infty} J'_m$, J'_m is open, for all m, and A is compact, there exist a finite collection of the J'_m, $1 \le m \le M$, covering A.

$$A \subset \bigcup_{m=1}^{M} J'_m.$$

This implies that if χ_A and $\chi_{J'_m}$ are the characteristic functions of A and J'_m respectively,

$$\chi_A \le \sum_{m=1}^{M} \chi_{J'_m}. \tag{1.3}$$

Now we appeal to the fact that the volume of an interval is the Riemann integral of its characteristic function. Since A is the union of closed intervals I_k, $1 \le k \le N$, with non-intersecting interiors, this implies that

$$\sum_{j=1}^{N} v(I_k) \le \sum_{m=1}^{M} v(J'_m). \tag{1.4}$$

Putting together equations (1.1), (1.2), (1.3) and (1.4) we get

$$\sum_{j=1}^{N} v(I_k) \le \sum_{m=1}^{M} v(J'_m) \le \sum_{m=1}^{M} (1+\epsilon) v(J_m) \le (1+\epsilon) \sum_{m=1}^{\infty} v(J_m) \le (1+\epsilon)^2 |A|_e.$$

Since this holds for every ϵ, it shows that $|A|_e \ge \sum_{j=1}^{N} v(I_j)$. This proves $P.2$ when the intervals are closed. To prove $P.2$ for pairwise disjoint open intervals, we first observe that

$$\bigcup_{j=1}^{N} I_j \subset \bigcup_{j=1}^{N} \overline{I}_j.$$

By $P.1$ and the first part of $P.2$,

$$\left|\bigcup_{j=1}^{N} I_j\right|_e \le \left|\bigcup_{j=1}^{N} \overline{I}_j\right|_e \le \sum_{j=1}^{N} v(I_j). \tag{1.5}$$

On the other hand, if $I_j = (a_1^j, b_1^j) \times ... \times (a_n^j, b_n^j)$, then for $0 < \delta < b_k^j - a_k^j$, $k = 1, ..., n$, $j = 1, ..., N$, et $I_j^* = [a_1^j + \delta, b_1^j - \delta] \times ... \times [a_n^j + \delta, b_n^j - \delta] \subset I_j$, and since the I_j are paiwise disjoint, the I_j^* are disjoint. Therefore by $P.1$, the first case of $P.2$, and (1.5)

$$\sum_{j=1}^{N} v(I_j^*) = \left|\bigcup_{j=1}^{N} I_j^*\right|_e \le \left|\bigcup_{j=1}^{N} I_j\right|_e \le \sum_{j=1}^{N} v(I_j).$$

Since this holds for every δ, $P.2$ also holds for open intervals.

Property $P.3$ is obvious if $|E_k|_e = \infty$ for some k. So we may assume $|E_k|_e < \infty$ for every k. Let $\epsilon > 0$ then for each k we can pick a collection $\{I_{j,k}, j \in \mathbb{N}\}$ of closed intervals such that

$$E_k \subset \bigcup_{j=1}^{\infty} I_{j,k}, \text{ and } |E_k|_e \le \sum_{j=1}^{\infty} v(I_{jk}) + \epsilon/2^k, \quad k = 1, 2, ... \tag{1.6}$$

Clearly, $\bigcup_{k=1}^{\infty} E_k \subset \bigcup_{j,k=1}^{\infty} I_{j,k}$, and hence

$$\left|\bigcup_{k=1}^{\infty} E_k\right|_e \le \sum_{j,k} v(I_{j,k})$$

Since this is a series of positive terms, we have from (1.6)

$$\left|\bigcup_{k=1}^{\infty} E_k\right|_e \le \sum_{j,k} v(I_{j,k}) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} v(I_{j,k}) \le \sum_{k=1}^{\infty} (|E_k|_e + \epsilon/2^k) \le \epsilon + \sum_{k=1}^{\infty} |E_k|_e.$$

Since ϵ is arbitrary, $P.3$ holds.

Since $|E|_e \le |\mathcal{O}|_e$ for every $E \subset \mathcal{O}$, $P.4$ is obvious if $|E|_e = \infty$. So we may assume that $|E|_e < \infty$.

For $\epsilon > 0$, let I_k, $k = 1, 2, ...$ be closed intervals such that

$$E \subset \bigcup_{k=1}^{\infty} I_k, \text{ and } \sum_{k=1}^{\infty} v(I_k) \leq |E|_e + \epsilon/2.$$

For each k, pick an open interval I'_k such that

$$I_k \subset I'_k \text{ and } v(I'_k) \leq v(I_k) + \epsilon/2^{k+1}. \tag{1.7}$$

Let $\mathcal{O} = \bigcup_{k=1}^{\infty} I'_k$. Then $\mathcal{O}$ is open and $E \subset \mathcal{O}$. By property $P.1$ $|E|_e \leq |\mathcal{O}|_e$. By property $P.3$, property $P.2$, applied to a single interval, and (1.7)

$$|\mathcal{O}|_e \leq \sum_{k=1}^{\infty} |I'_k|_e = \sum_k v(I'_k) \leq \sum_k (v(I_k) + \epsilon/2^{k+1}) \leq \epsilon/2 + \sum_k v(I_k) \leq |E|_e + \epsilon.$$

Thus for every $\epsilon > 0$ there exists an open subset $\mathcal{O}$ such that $E \subset \mathcal{O}$ and that $|E|_e \leq |\mathcal{O}|_e \leq |E| + \epsilon$. This proves $P.4$.

Now we prove $P.5$. If $|E|_e = \infty$, just take $H = \mathbb{R}^n$. So we may assume that $|E|_e < \infty$. By property $P.4$, there exists a sequence of open subsets $\mathcal{O}_k$, $k = 1, 2, ...$, such that $E \subset \mathcal{O}_k$ and

$$|\mathcal{O}_k|_e \leq |E|_e + 1/k.$$

Let $H \bigcap_{k=1}^{\infty} \mathcal{O}_k$. Then H is a G_δ, $E \subset H \subset \mathcal{O}_k$, $k = 1, 2, ...$ Therefore, by property $P.1$

$$|E|_e \leq |H|_e \leq |\mathcal{O}_k| \leq |E_e| + 1/k, \quad k = 1, 2, ...$$

Thus $|H|_e = |E|_e$.

This ends the proof of the theorem. □

The following is an important generalization of Property $P.2$

Proposition 1.1. *Let $E_1 \subset \mathbb{R}^n$ and $E_2 \subset \mathbb{R}^n$ be such that $d(E_1, E_2) > 0$, where*

$$d(E_1, E_2) = \inf\{|x - x'| : x \in E_1, x' \in E_2\}.$$

Then $|E_1 \cup E_2|_e = |E_1|_e + |E_2|_e$.

Proof. If either $|E_1|_e = \infty$ or $|E_2|_e = \infty$, the statement is true. So we may assume that $|E_1|_e < \infty$ and $|E_2|_e < \infty$. From property $P.3$ we know that

$$|E_1 \cup E_2|_e \leq |E_1|_e + |E_2|_e.$$

So we only need to prove the opposite inequality. Since $|E_1 \bigcup E_2|_e < \infty$, for any $\epsilon > 0$ there exists a cover of $E_1 \bigcup E_2$ by closed intervals $\{I_k, \ k \in \mathbb{N}\}$ such that

$$\sum_{k=1}^{\infty} v(I_k) \leq \left| E_1 \bigcup E_2 \right|_e + \epsilon.$$

We assume that for every interval I_k in this cover $I_k \cap (E_1 \cup E_2) \neq \emptyset$, otherwise we can just remove it from the collection. We then divide this family into three parts:

$$\begin{gathered}
\{I_k, \quad k \in \mathbb{N}\} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3, \text{ where} \\
I_k \in \mathcal{F}_1 \text{ if and only if } I_k \cap E_1 \neq \emptyset, \quad I_k \cap E_2 = \emptyset, \\
I_k \in \mathcal{F}_2 \text{ if and only if } I_k \cap E_1 = \emptyset, \quad I_k \cap E_2 \neq \emptyset, \\
I_k \in \mathcal{F}_3 \text{ if and only if } I_k \cap E_1 \neq \emptyset, \quad I_k \cap E_2 \neq \emptyset.
\end{gathered}$$

Each interval in $I_k \in \mathcal{F}_3$ can be divided into a finite number of intervals $I_{j,k}$ with diameter less than $d(E_1, E_2)$. And we have

$$v(I_k) = \sum v(I_{j,k}). \tag{1.8}$$

But each interval $I_{j,k}$ falls into the family $\mathcal{F}_1$ or $\mathcal{F}_2$, or it does not intersect $E_1 \cup E_2$. We throw out the intervals that do not intersect $E_1 \cup E_2$. Then we replace each of the intervals $I_k \in \mathcal{F}_3$ by the intervals in $\{I_{j,k}\}$ which are in $\mathcal{F}_1$ or $\mathcal{F}_2$. We have now a family of closed intervals $\{I'_k\} = \mathcal{F}_1 \cup \mathcal{F}_2$ which cover $E_1 \cup E_2$, and consists of the intervals $\{I_k\}$ which are in either $\mathcal{F}_1$ or $\mathcal{F}_2$, and the intervals in $\{I_{jk}\}$ which are also in $\mathcal{F}_1 \cup \mathcal{F}_2$. In view of (1.8)

$$\sum_{k=1}^{\infty} v(I'_k) \leq \left| E_1 \bigcup E_2 \right|_e + \epsilon. \tag{1.9}$$

Therefore, by definition of outer measure

$$|E_1|_e + |E_2|_e \leq \sum_{I'_k \in \mathcal{F}_1} v(I'_k) + \sum_{I'_k \in \mathcal{F}_2} v(I'_k) = \sum_k v(I'_k) \leq |E_1 \cup E_2|_e + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this implies that

$$|E_1|_e + |E_2|_e \leq \left| E_1 \bigcup E_2 \right|_e.$$

□

So far we have constructed the "outer measure," which is defined for all subsets of $\mathbb{R}^n$. But we do not know whether the "outer measure" is in fact a measure, i.e. if it is σ-additive. In fact it is not.

Proposition 1.2. *There exists a family $\{A_k \subset \mathbb{R} : k = 0, 1, ...\}$ such that $A_k \cap A_j = \emptyset$ and $|\bigcup_{k=1}^{\infty} A_k|_e < \sum_{k=1}^{\infty} |A_k|_e$.*

Proof. The first thing we need to observe is that if $A \subset \mathbb{R}^n$ and $w \in \mathbb{R}^n$, then $|A + w|_e = |A|_e$. Indeed, if $\{I_k, k = 1, ...\}$ are closed intervals that cover A, then the translates $\{I_k + w\}$ will cover $A + w$. Reciprocally, if $\{J_k,\ k \in \mathbb{N}\}$, cover of $A + w$ then $\{J_k - w,\ \ k \in \mathbb{N}\}$ cover A. So it follows from the definition of outer measure, and the invariance of the volume of an interval under translation, that $|A|_e = |A + w|_e$.

Next we define the following relation for elements of $I = [-1/2, 1/2]$: we say tat $x \sim y$ if $x - y \in \mathbb{Q} \cap [-1, 1]$. It is easy to see that $\sim$ is an equivalence relation. Let $A = [-1/2, 1/2]/\sim = \{[x] : x \in [-1/2, 1/2]\}$, where $[x]$ denotes the class of equivalence of the point x. Let r_k, $k \in \mathbb{N}$ be an enumeration of the rational numbers in $[-1, 1]$, and for each k, let $A_k = A + r_k$. Then $A_k \cap A_j = \emptyset$ if $k \neq j$. Otherwise $[x] + r_j = [y] + r_k$ and then $[x] - [y] = r_k - r_j \in \mathbb{Q}$. This implies that $[x] = [y]$ but then $r_j = r_k$.

One one hand $A_k \subset [-3/2, 3/2]$ for $k = 1, 2, ...$, therefore $\bigcup_{k=1}^{\infty} A_k \subset [-3/2, 3/2]$. On the other hand, by the definition of the equivalence relation, $[-1/2, 1/2] \subset \bigcup_{k=1}^{\infty} A_k$. So we conclude that

$$[-1/2, 1/2] \subset \bigcup_{k=1}^{\infty} A_k \subset [-3/2, 3/2].$$

Therefore

$$1 \leq \left| \bigcup_{k=1}^{\infty} A_k \right|_e \leq 3. \tag{1.10}$$

But we also know that $|A_k|_e = |A|_e$. If $|\bigcup_{k=1}^{\infty} A_k|_e = \sum_{k=1}^{\infty} |A_k|_e$, it would follow that $|A_k|_e = 0$, but then, by property $P.3$, $|\bigcup_{k=1}^{\infty} A_k|_e = 0$, which contradicts (1.10). □

To make the "outer measure" σ-additive, we have to restrict the σ-algebra where it is defined.

Definition 1.1. *Let $\mathcal{L}$ be the family of subsets $E \subset \mathbb{R}^n$ such that for any $\epsilon > 0$ there exists an open subset $\mathcal{O} \subset \mathbb{R}^n$ such that*

$$E \subset \mathcal{O} \text{ and } |\mathcal{O} \setminus E|_e < \epsilon.$$

If $E \in \mathcal{L}$ we say that E is Lebesgue measurable.

This is the class of sets we will work with.

Theorem 1.2. *The family $\mathcal{L}$ is a σ-algebra and the function*

$$\mu : \mathcal{L} \longrightarrow [0, \infty]$$
$$\mu(E) = |E|_e$$

is a positive measure on $\mathcal{L}$.

The proof of this theorem will be divided into several lemmas.

Lemma 1.2. *If $E \subset \mathbb{R}^n$ is an open subset, then $E \in \mathcal{L}$. If $E \subset \mathbb{R}^n$ is such that $|E|_e = 0$, then $E \in \mathcal{L}$.*

Proof. Of course, if E is open one can take $\mathcal{O} = E$ and thus E satisfies the requirement of the definition.

If $|E|_e = 0$, then it follows from property $P.4$ that for any $\epsilon > 0$, there exists an open set $\mathcal{O}$ such that $E \subset \mathcal{O}$ and $|\mathcal{O}|_e < \epsilon$. Since $\mathcal{O} \setminus E \subset \mathcal{O}$, property $P.1$ implies that $|\mathcal{O} \setminus E|_e < \epsilon$. □

Lemma 1.3. *Let $E_k \in \mathcal{L}$, $k = 1, 2, ...$ then $E = \bigcup_{k=1}^{\infty} E_K \in \mathcal{L}$.*

Proof. Let $\epsilon > 0$. Since $E_k \in \mathcal{L}$, there exists $\mathcal{O}_k$, open, such that

$$E_k \subset \mathcal{O}_k \text{ and } |\mathcal{O}_k \setminus E_k|_e < \epsilon/2^k, \quad k = 1, 2, ...$$

Hence $\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{O}_k \supset E$. Moreover,

$$\mathcal{O} \setminus E = \bigcup_{k=1}^{\infty} (\mathcal{O}_k \setminus E) \subset \bigcup_{k=1}^{\infty} (\mathcal{O}_k \setminus E_k).$$

Therefore, in view of property $P.3$,

$$|\mathcal{O} \setminus E|_e \leq \sum_{k=1}^{\infty} |\mathcal{O}_k \setminus E_k|_e < \epsilon.$$

This ends the proof of the Lemma. □

Lemma 1.4. *If $F \subset \mathbb{R}^n$ is closed, then $F \in \mathcal{L}$.*

Proof. We will show that if F is compact then $F \in \mathcal{L}$. For a general closed set F, we write $F = \bigcup_{k=1}^{\infty} F_k$, $F_k = F \cap \{x : |x| \leq k\}$, which is compact. Then by Lemma 1.3, $F \in \mathcal{L}$.

So let F be compact. Then for $\epsilon > 0$ there exists an open set $\mathcal{O}$ such that $F \subset \mathcal{O}$ and $|\mathcal{O}|_e \leq |F|_e + \epsilon$. We claim that in fact $|\mathcal{O} \setminus F|_e < \epsilon$.

Since $\mathcal{O} \setminus F$ is open, then by Lemma 1.1,

$$\mathcal{O} \setminus F = \bigcup_{j=1}^{\infty} I_j, \quad I_j \text{ non-overlapping closed intervals.}$$

Hence

$$|\mathcal{O} \setminus F|_e \leq \sum_{j=1}^{\infty} v(I_j).$$

On the other hand

$$\mathcal{O} = F \cup \left(\bigcup_{j=1}^{\infty} I_j \right) \supset F \cup \left(\bigcup_{j=1}^{N} I_j \right), \quad \forall \ \ N \in \mathbb{N}.$$

Hence

$$|\mathcal{O}|_e \geq \left| F \cup \left(\bigcup_{j=1}^{N} I_j \right) \right|_e, \quad N \in \mathbb{N}. \tag{1.11}$$

Since F and $\bigcup_{j=1}^{N} I_j$ are compact disjoint subsets, the distance between them is positive, and then Proposition 1.1 guarantees that

$$\left| F \cup \bigcup_{j=1}^{N} I_j \right|_e = |F|_e + \left| \bigcup_{j=1}^{N} I_j \right|_e.$$

However, since the intervals I_k are non-overlapping property $P.2$ implies that $\left|\bigcup_{j=1}^{N} I_j\right|_e = \sum_{k=1}^{N} v(I_k)$. Hence

$$\left| F \cup \bigcup_{j=1}^{N} I_j \right|_e = |F|_e + \sum_{k=1}^{N} v(I_k).$$

So we conclude from (1.11) that

$$|F|_e + \sum_{k=1}^{N} v(I_k) \leq |\mathcal{O}|_e, \quad N = 1, 2, ...$$

Therefore

$$\sum_{k=1}^{N} v(I_k) \leq |\mathcal{O}|_e - |F|_e < \epsilon, \quad N = 1, 2, ...$$

So

$$|\mathcal{O} \setminus F|_e \leq \sum_{k=1}^{\infty} v(I_k) \leq \epsilon.$$

This concludes the proof of the lemma. □

Lemma 1.5. *Suppose $E \in \mathcal{L}$ then $\mathbb{R}^n \setminus E \in \mathcal{L}$.*

Proof. Let $\mathcal{O}_k$ be open subsets of $\mathbb{R}^n$, $k = 1, 2, ...$ with $E \subset \mathcal{O}_k$ and

$$|\mathcal{O}_k \setminus E|_e \leq 1/k, \quad k = 1, 2, ...$$

Now $\mathbb{R}^n \setminus \mathcal{O}_k$ is closed, and therefore in $\mathcal{L}$. Moreover

$$\mathbb{R}^n \setminus \mathcal{O}_k \subset \mathbb{R}^n \setminus E, \quad k = 1, 2, ...$$

Therefore

$$H = \bigcup_{k=1}^{\infty} (\mathbb{R}^n \setminus \mathcal{O}_k) \subset \mathbb{R}^n \setminus E$$

H is an F_σ, and thus $H \in \mathcal{L}$. Let $A = (\mathbb{R}^n \setminus E) \setminus H$. Then

$$\mathbb{R}^n \setminus E = H \cup A.$$

We claim that $|A|_e = 0$. In view of Lemma 1.2, this shows that $\mathbb{R}^n \setminus E \in \mathcal{L}$.

To prove our claim, notice that

$$A = (\mathbb{R}^n \setminus E) \setminus \left(\bigcup_{k=1}^{\infty} (\mathbb{R}^n \setminus \mathcal{O}_k) \right) \subset (\mathbb{R}^n \setminus E) \setminus (\mathbb{R}^n \setminus \mathcal{O}_k) = \mathcal{O}_k \setminus E.$$

Hence $A \subset \mathcal{O}_k \setminus E$, $k = 1, 2, ...$ Then

$$|A|_e \leq |\mathcal{O}_k \setminus E|_e \leq 1/k, \quad k = 1, 2, ...$$

Hence $|A|_e = 0$. □

This Lemma has an important consequence:

Corollary 1.1. *If $E \in \mathcal{L}$, then $E = H \cup A$, H an F_σ and $|A|_e = 0$.*

Proof. The proof of the Lemma 1.5 shows that if $E \in \mathcal{L}$, then $\mathbb{R}^n \setminus E = H \cup A$, where H is an F_σ and $|A|_e = 0$. Just apply this result to $E = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus E))$. □

Another important consequence of this is

Corollary 1.2. *If $E \in \mathcal{L}$ then for every $\epsilon > 0$, there exists a closed set $F \subset \mathbb{R}^n$ such that $F \subset E$ and $|E \setminus F|_e < \epsilon$.*

Finally we prove that the outer measure restricted to $\mathcal{L}$ gives a measure

Theorem 1.3. *The function*

$$\mu : \mathcal{L} \longrightarrow [0, \infty]$$
$$\mu(E) = |E|_e$$

is a measure.

Proof. Let $E_j \in \mathcal{L}$, $j = 1, 2, ...$ be such that $E_j \cap E_k = \emptyset$ if $j \neq k$. We know from property $P.3$ that

$$\mu \left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} \mu(E_j).$$

We want to show that

$$\sum_{j=1}^{\infty} \mu(E_j) \leq \mu \left(\bigcup_{j=1}^{\infty} E_j \right). \tag{1.12}$$

Assume that E_j is bounded $j = 1, 2, ...$ Given $\epsilon > 0$, by Corollary 1.2 there exists a closed set $F_k \subset E_k$ such that $\mu(E_k \setminus F_k) < \epsilon/2^k$. Since

$$E_k = F_k \cup (E_k \setminus F_k),$$

we have that

$$\mu(E_k) \leq \mu(F_k) + \epsilon/2^k. \tag{1.13}$$

The family $\{F_k\}$ consists of pairwise disjoint compact subsets. Therefore, for any $N \in \mathbb{N}$,

$$\mu \left(\bigcup_{k=1}^{N} F_k \right) = \sum_{k=1}^{N} \mu(F_k) \leq \mu(E), \quad N = 1, 2, ...$$

Hence

$$\sum_{k=1}^{\infty} \mu(F_k) \leq \mu(E). \tag{1.14}$$

So we conclude from (1.13) and (1.14) that

$$\sum_{k=1}^{\infty} \mu(E_k) \leq \sum_{k=1}^{\infty} \mu(F_k) + \epsilon/2^k \leq \mu(E) + \epsilon.$$

Since this holds for every ϵ (1.12) must hold.

In general, if E_k is not bounded, let $I_0 = \emptyset$ and $I_j = [-j,j] \times ... \times [-j,j]$, $j = 1, 2, ...$ Let $S_j = I_j \setminus I_{j-1}$, $j = 1, 2, ...$ Then $E_{k,j} = E_k \cap S_j$ are measurable, pairwise disjoint and bounded. Therefore

$$\mu(E_k) = \sum_{j=1}^{\infty} \mu(E_{j,k}).$$

Moreover

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} E_{j,k}$$

So

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{j,k=1}^{\infty} E_{j,k}\right) = \sum_{j,k=1}^{\infty} \mu(E_{j,k}) = \sum_{k=1}^{\infty} \mu(E_k).$$

Therefore μ is σ-additive, and we are done. □

We end this part with a very important characterization of the Lebesgue measurable sets.

Theorem 1.4. *(Carathéodory) A subset $E \subset \mathbb{R}^n$ is Lebesgue measurable if and only if, for every $A \subset \mathbb{R}^n$*

$$|A|_e = |A \cap E|_e + |A \setminus E|_e \tag{1.15}$$

Proof. Suppose first that $E \in \mathcal{L}$ and we will prove (1.15). Let $A \subset \mathbb{R}^n$. Since $A = A \cap E \cup (A \setminus E)$, we have

$$|A|_e \leq |A \cap E|_e + |A \setminus E|_e. \tag{1.16}$$

We know that there exists a G_δ set H such that $H \supset A$ and $|H|_e = |A|_e$. Since $H \in \mathcal{L}$, and $H = H \cap E \cup (H \setminus E)$, is a union of disjoint measurable sets. Then

$$|A|_e = |H|_e = |H \cap E|_e + |H \setminus E|_e$$

But $H \cap E \supset A \cap E$ and $H \setminus E \supset A \setminus E$ therefore

$$|A|_e = |H|_e = |H \cap E|_e + |H \setminus E|_e \geq |A \cap E|_e + |A \setminus E|_e. \tag{1.17}$$

Then (1.16) and (1.17) show (1.15).

Now suppose (1.15) holds for every A. Let us first make the extra assumption that $|E|_e < \infty$. In that case, there exists a G_δ set H such that

$$E \subset H \text{ and } |H|_e = |E|_e.$$

But $H \in \mathcal{L}$ and using (1.15) with $A = H$ we find that

$$|H|_e = |H \cap E|_e + |H \setminus E|_e = |E|_e + |H \setminus E|.$$

Hence $|H \setminus E|_e = 0$. Therefore $H \setminus E \in \mathcal{L}$ Since $E = H \setminus (H \setminus E)$ it follows that $E \in \mathcal{L}$.

In the case where $|E|_e = \infty$, write $E = \bigcup_{k=1}^{\infty} (E \cap \{x : |x| < k\})$.

Let H_k be a G_δ set such that

$$E_k \subset H_k, \quad |H_k|_e = |E_k|_e.$$

Applying (1.15) to H_k we have

$$|E_k|_e = |H_k|_e = |H_k \cap E|_e + |H_k \setminus E|_e$$

But $H_k \cap E \supset H_k \cap E_k = E_k$ and $H_k \setminus E \subset H_k \setminus E_k$. So

$$|H_k \cap E|_e \geq |E_k|_e, \quad |H_k \setminus E|_e = 0.$$

Therefore $H_k \setminus E \in \mathcal{L}$ and hence, if $H = \bigcup_{k=1}^{\infty} H_k$, $H \in \mathcal{L}$, and $E \subset H$. But $H \setminus E = \bigcup_{k=1}^{\infty}(H_k \setminus E) \in \mathcal{L}$ and therefore $E = H \setminus (H \setminus E) \in \mathcal{L}$. □