Let $(X, \mathcal{T}, \mu)$; $(Y, \mathcal{T}, \nu)$ be σ- finite measure spaces.

<u>Theorem:</u> If either $f: X\times Y \longrightarrow [0,\infty]$ is measurable or $f \in L^1(X\times Y, \mu\times\nu)$, then $f: X\times Y \longrightarrow \mathbb{C}$ is measurable and

$$\int_{X\times Y} f(x,y)\, d(\mu\times\nu) = \int_X \left(\int_Y f(x,y)\, d\nu \right) d\mu = \int_Y \left(\int_X f(x,y)\, d\mu \right) d\nu.$$

<u>Remark:</u> The result does not hold if $f \notin L^1(X\times Y, \mu\times\nu)$.

Example: $f(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$ $\quad x, y \in (0,1)$.

$$\int_0^1 \int_0^1 |f(x,y)|\, dx\, dy \geqslant \int_{\Omega} |f(x,y)|\, dx\, dy$$

$$\Omega = \{ 0 \le x^2 + y^2 \le 1 \ ; \ \tan\theta < 1/2 \}$$

$$= \int_{\Omega} \frac{\cos 2\theta}{r}\, dr\, d\theta = \infty.$$

But $$\int_0^1 f(x,y)\, dy = \frac{1}{x} \int_0^{\tan^{-1}(1/x)} \cos 2\theta = \frac{1}{1+x^2}$$

$y = x \tan\theta$

Obviously, by interchanging x and y, we have

$$\int_0^1 f(x,y)\,dx = -\frac{1}{1+y}$$

So $$\int_0^1\left(\int_0^1 f(x,y)\,dy\right)dx = -\int_0^1\left(\int_0^1 f(x,y)\,dx\right)dy = \pi/4$$

Let $(\mathcal{L}_k, \mu_k)$ denote the Lebesgue σ-algebra and measure on $\mathbb{R}^k$. Suppose $k = r+s$, so $\mathbb{R}^k \sim \mathbb{R}^r \times \mathbb{R}^s$. Is $\mathcal{L}_r \times \mathcal{L}_s = \mathcal{L}_k$ and $\mu_r \times \mu_s = \mu_k$?

The answer is no. ~~and~~, Let $A \in \mathcal{L}_r$; $A \neq \emptyset$ and $\mu(A) = 0$. Let $B \in \mathcal{L}_s$. Now let $C \subset B$, $C \notin \mathcal{L}_s$. Then

$$A \times C \subset A \times B$$ and since

$$\mu_r \times \mu_s (A \times B) = \mu_r(A) \cdot \mu_s(B) = 0$$

then if $\mu_r \times \mu_s = \mu_k$, $A \times C$ would be measurable.

However if $A \times C \in \mathcal{L}_r \times \mathcal{L}_s$, then if $a \in A$

$$(A \times C)_a = C \in \mathcal{L}_s$$ which is a contradiction.

Completion of measures:

<u>Theorem:</u> Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{M}^*$ be the family of all $E \subset X$ for which there exist $A, B \in \mathcal{M}$ such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. Define $\mu^*(E) = \mu(A)$. Then $(X, \mathcal{M}^*, \mu^*)$ is a measure space. μ^* is the completion of μ.

<u>Proof:</u>

(1) $\mathcal{M} \subset \mathcal{M}^*$. If $E \in \mathcal{M}$, take $A = B = E$.

In this case $X \in \mathcal{M}^*$

(2) Let $E \in \mathcal{M}^*$, then $\exists$ $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. In this case $B^c \subset E^c \subset A^c$ and

$\mu(A^c \setminus B^c) = \mu(B \setminus A) = 0$. So $E^c \in \mathcal{M}^*$.

(3) If $E_j \in \mathcal{M}^*$, $i = 1, 2, \ldots$ Then there exist $A_j, B_j \in \mathcal{M}$ with $A_j \subset E_j \subset B_j$ and $\mu(B_j \setminus A_j) = 0$

Then $A = \bigcup_{j=1}^{\infty} A_j \subset \bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty} B_j = B$

$B \setminus A \subset \bigcup_{j=1}^{\infty} (B_j \setminus A_j)$ and $\mu(B \setminus A) = 0$.

(4) μ^* is well defined.

If $A \subset E \subset B$ and $\tilde{A} \subset E \subset \tilde{B}$ with

$\mu(B \setminus A) = \mu(\tilde{B} \setminus \tilde{A}) = 0$, then

$A \setminus \tilde{A} \subset \tilde{B} \setminus \tilde{A}$ and $\tilde{A} \setminus A \subset B \setminus A$. So

$\mu(A \setminus \tilde{A}) = \mu(\tilde{A} \setminus A) = 0$. Since

$A = (A \cap \tilde{A}) \cup (A \setminus \tilde{A})$ and $\tilde{A} = (A \cap \tilde{A}) \cup (\tilde{A} \setminus A)$,

we have $\mu(A) = \mu(\tilde{A}) = \mu(A \cap \tilde{A})$.

(5) μ^* is a measure.

Let $E_j \in m^*$; $j = 1, 2, \cdots$, $E_j \cap E_k = \phi$.

Let $A_j, B_j \in m$ with $A_j \subset E_j \subset B_j$, $\mu(B_j \setminus A_j) = 0$.

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \le \mu^*\left(\bigcup_{j=1}^{\infty}\left(A_j \cup (B_j \setminus A_j)\right)\right)$$

$$\le \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) + \mu^* \bigcup_{j=1}^{\infty}(B_j \setminus A_j) \le \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right)$$

$$= \sum_{j=1}^{\infty} \mu(A_j) \underset{\uparrow}{=} \sum_{j=1}^{\infty} \mu^*(E_j).$$

By definition

Theorem: If $k = r+s$, then $(\mathcal{L}_k, \mu_k)$ is the completion of $(\mathcal{L}_r \times \mathcal{L}_s ; \mu_r \times \mu_s)$.

Proof: Let $m = (\mathcal{L}_r \times \mathcal{L}_s)^*$ and $\nu = (\mu_r \times \mu_s)^*$

Let $I = \prod_{j=1}^{k} [a_j, b_j] = \prod_{j=1}^{r} [a_j, b_j] \times \prod_{j=r+1}^{k} [a_j, b_j]$

So I is a measurable rectangle and hence $I \in \mathcal{L}_r \times \mathcal{L}_s$. Therefore open subsets are contained in $\mathcal{L}_r \times \mathcal{L}_s$.

Also $\mu_k(I) = (\mu_r \times \mu_s)(I)$

and $\mu_r \times \mu_s$ is invariant under translation.

So $(\mu_r \times \mu_s)^*$ is also invariant under translation.

So $(\mu_r \times \mu_s)^* = \mu_k$.

Convolutions

Let $f \in L^1(\mathbb{R}^n)$ and $g \in C_c(\mathbb{R}^n)$, define

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

Let $z = x - y$

$$f * g(x) = \int f(z)\, g(x-z)\, dz = g * f(x).$$

<u>Proposition:</u> $f * g \in C(\mathbb{R}^n)$.

Let $x_j \to x$

$$f * g(x_j) = \int g(x_j - z)\, f(z)\, dz$$

by the dominated convergence theorem

$$\lim_{j\to\infty} f * g(x_j) = \int g(x-z)\, f(z)\, dz = g * f.$$

<u>Natural Question:</u> Can this operation be defined for $f, g \in L^1(\mathbb{R}^n)$. ?

<u>Theorem:</u> Let $f, g \in L^1(\mathbb{R}^n)$. Then

$$(*) \quad \int |f(x-y)\, g(y)|\, dy < \infty \quad \text{a.e. } x$$

Let $h(x) = \int f(x-y)\, g(y)\, dy$ if $(*)$ holds for x, and $h(x) = 0$ otherwise. Then $h \in L^1(\mathbb{R}^n)$

and

$$\|h\|_{L^1} \leq \|f\|_{L^1} \cdot \|g\|_{L^1}.$$

<u>Proof:</u>