Convolutions :

Theorem: Let $f, g \in L^1(\mathbb{R}^n)$. Then

$$(*) \quad \int |f(x-y)\, g(y)|\, dy < \infty \quad \text{a.e}$$

Let $h(x) = \int f(x-y)\, g(y)\, dy$ if $*$ is satisfied

and $h(x) = 0$ otherwise. Then $h \in L^1(\mathbb{R}^n)$ and.

$$\|h\|_{L^1} \leq \|f\|_{L^1} \cdot \|g\|_{L^1}.$$

Proof.: Let $\psi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, then if
$(x, y) \longmapsto y$

$L \subset \mathbb{R}^n$ is Lebesgue measurable, $\psi^{-1}(L) = \mathbb{R}^n \times L$ is Lebesgue measurable. Let

$$\varphi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$(x, y) \longmapsto x - y$$

Notice that $\varphi = \psi \circ \eta$ where

$$\eta : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$
$$(x, y) \longmapsto (x+y, x-y).$$

Therefore $\tilde{f}(x,y) = f(x-y) = f \circ \varphi$ and $\tilde{g} = g \circ \psi$ are Lebesgue measurable in $\mathbb{R}^n \times \mathbb{R}^n$. To see that Let $V \subset \mathbb{C}$ be open. Then $\tilde{f}^{-1}(V) = \varphi^{-1}(f^{-1}(V))$ is Lebesgue measurable. Similarly $\tilde{g}^{-1}(V) = \psi^{-1}(g^{-1}(V))$ is also Lebesgue measurable.

Therefore $f(x-y)\,g(y)$ is Lebesgue measurable.

Next

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} | f(x-y)\, g(y) | \, dy \right) dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|\, |g(y)| \, dx \right) dy$$

$$= \|f\|_1 \cdot \|g\|_1 .$$

Hence $f(x-y)\,g(y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ by Fubini's theorem. Again, by Fubini's theorem

$$h(x) = \int f(x-y)\, g(y)\, dy \quad \text{is defined a.e.}$$

One can generalize this theorem as follows:

<u>Theorem</u>: If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, and $g \in L^1(\mathbb{R}^n)$ then

$$\|f * g\|_{L^p} \le \|f\|_p \cdot \|g\|_1 .$$

Proof: Let $F(x) = \int |f(x-y)| \cdot |g(y)| \, dy$

In this case one can write:

$$F(x)^p = \|g\|_1 \int_{\mathbb{R}^n} |f(x-y)| \cdot \frac{|g(y)|}{\|g\|_1} \, dy$$

Since $\int_{\mathbb{R}^n} \frac{|g(y)|}{\|g\|_1} \, dy = 1$, Jensen's inequality gives that

$$F(x)^p \leq \|g\|_1^p \cdot \int |f(x-y)|^p \cdot \frac{|g(y)|}{\|f\|_1} \, dy$$

$$= \|g\|_1^{p-1} \cdot \int |f(x-y)|^p \, |g(y)| \, dy$$

Therefore:

$$\int_{\mathbb{R}^n} F(x)^p \, dx \leq \|g\|_1^{p-1} \cdot \int \left(\int |f(x-y)|^p \cdot |g(y)| \, dy \right) dx$$

$$= \|g\|_1^{p-1} \cdot \|f\|_p^p \cdot \|g\|_1 = \|g\|_1^p \cdot \|f\|_p^p$$

Therefore $\|f * g\|_p \leq \|g\|_1 \cdot \|f\|_p$.

C_0^∞ functions with Compact Support

Let $\chi(x) = \begin{cases} e^{-1/x} ; & x > 0 \\ 0 ; & x \leq 0 \end{cases}$

<u>Claim.</u> $\chi(x)$ is C^∞ for every $x \in \mathbb{R}$. The only possible problem is at $x = 0$. To check that $\chi(x)$ is C^∞ there just notice that for $x > 0$

$$\chi^{(k)}(x) = p_k(1/x)\, e^{-1/x}, \quad x > 0$$

where p_k is a polynomial., and therefore

$$\lim_{x \to 0^+} \chi^{(k)}(x) = 0$$

Let $z \in \mathbb{R}^n$ and define

$$\tilde{\chi}(z) = \chi\left(1 - |z|^2\right)$$

$\tilde{\chi}(z)$ is a C^∞ function and $\tilde{\chi}(z) = 0$ if $|z| \geq 1$. It is convenient to work with $\chi^*(z) = \dfrac{1}{\|\tilde{\chi}\|_{L^1}} \cdot \tilde{\chi}$. So $\chi^* \in C_0^\infty(\mathbb{R}^n)$ and $\|\chi^*\|_{L^1} = 1$, $\chi^* \geq 0$.

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$; $\int \varphi(x)\,dx = 1$, $\varphi(x) = 0$ if $|x| \geq 1$. ~~and~~ and $\varphi(x) \geqslant 0$

$$\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon).$$ Then $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, $\varphi(x) = 0$ if $|x| \geqslant \varepsilon$ and $\int \varphi_\varepsilon(x)\,dx = 1$.

Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ and define.

$$f_\varepsilon(x) = f * \varphi_\varepsilon(x) = \varphi_\varepsilon * f(x)$$

<u>Theorem</u>: $f_\varepsilon \in C^\infty(\mathbb{R}^n)$ and $\|f_\varepsilon - f\|_p \to 0$ as $\varepsilon \to 0$.

<u>Proof</u>: Let $x_j \to x$, then, fixed ε,

$$f_\varepsilon(x_j) = \int \varphi_\varepsilon(x_j - y)\, f(y)\,dy$$

We want to show that $f_\varepsilon(x_j) \to f_\varepsilon(x)$ as $x_j \to x$.

If $f \in L^1(\mathbb{R}^n)$; this is a consequence of the dominated convergence theorem. If $p > 1$.

$$|f_\varepsilon(x_j) - f_\varepsilon(x)| \leq \int \left| \varphi_\varepsilon(x_j - y) - \varphi_\varepsilon(x - y) \right| |f(y)|\,dy$$
$$\leq \|f\|_p \cdot \|\varphi_\varepsilon(x_j - y) - \varphi_\varepsilon(x - y)\|_{L^q}.$$

Therefore $f_\varepsilon(x)$ is continuous. To see that it is differentiable.

$$\frac{1}{h}\left(f_\varepsilon(x+h)-f_\varepsilon(x)\right)=\int \frac{1}{h}\left(\varphi_\varepsilon(x+h-y)-\varphi_\varepsilon(x-y)\right) f(y)\,dy$$

Again, if $f \in L^1(\mathbb{R}^n)$, the dominated convergence theorem gives that

$$\partial_{x_p} f_\varepsilon(x) = (\partial_{x_p}\varphi_\varepsilon) * f.$$

When $f \in L^p(\mathbb{R}^n)$, $p>1$, we write for $h=(1,0,\dots,0)$

$$\frac{1}{h}\left(f_\varepsilon(x+h)-f_\varepsilon(x)\right)-(\partial_{x_1}\varphi_\varepsilon) * f(x) = D(h)$$

$$=\int\left[\frac{1}{h}\left(\varphi_\varepsilon(x+h-y)-\varphi_\varepsilon(x-y)\right)-\partial_{x_1}\varphi_\varepsilon(x-y)\right] f(y)\,dy$$

Then

$$D(h) \leq \|f\|_{L^p}\cdot\left\|\frac{1}{h}\left(\varphi_\varepsilon(x+h)-\varphi_\varepsilon(x-y)\right)-\partial_{x_1}\varphi_\varepsilon(x-y)\right\|_{L^q}$$

$$\lim_{h\to 0} D(h)=0. \quad \blacksquare$$

To see that $f_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$ we write.

$$f(x) - f_\varepsilon(x) = f(x) - \int f(x-y)\,\varphi_\varepsilon(y)\,dy =$$

$$= \int \big(f(x) - f(x-y)\big)\,\varphi_\varepsilon(y)\,dy.$$

Since $\varphi_\varepsilon(y) = \varepsilon^{-n}\varphi(y/\varepsilon)$ we obtain, by setting $y = \varepsilon z$

$$f(x) - f_\varepsilon(x) = \int \big(f(x) - f(x-\varepsilon z)\big)\,\varphi(z)\,dz$$

If $f \in C^0(\mathbb{R}^n)$, since the integration is over $|z| \le 1$, we would have.

$$\|f - f_\varepsilon\|_{L^p} \le \left[\int_{|z|\le 1} |f(x) - f(x-\varepsilon z)|^p\,dz\right]^{1/p} \cdot \|\varphi\|_{L^q}$$

Therefore $\lim_{\varepsilon \to 0} \|f - f_\varepsilon\|_{L^p} = 0$, if $f \in C^0(\mathbb{R}^n)$.

In general, for $\delta > 0$, pick $g \in C^0_c(\mathbb{R}^n)$ so that $\|f - g\|_{L^p} < \delta$. and write

$$g_\varepsilon = g * \varphi_\varepsilon$$

$$\| f_\varepsilon - f \|_{L^p} = \| f_\varepsilon - g_\varepsilon + g_\varepsilon - g + g - f \|_{L^p}$$

$$\leq \| f - g \|_{L^p} + \| g_\varepsilon - g \|_{L^p} + \| f_\varepsilon - g_\varepsilon \|_{L^p}$$

Notice that $f_\varepsilon - g_\varepsilon = (f-g) * \varphi_\varepsilon$ and hence

$$\| f_\varepsilon - g_\varepsilon \|_{L^p} \leq \| f - g \|_{L^p} \cdot \| \varphi_\varepsilon \|_{L^1} = \| f - g \|_{L^p}.$$

Therefore

$$\| f_\varepsilon - f \|_{L^p} \leq 2\delta + \| g_\varepsilon - g \|_{L^p}.$$

Taking the limit as $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon \to 0} \| f_\varepsilon - f \|_{L^p} \leq 2\delta, \quad \forall \delta > 0. \quad \square.$$