Theorem Let $f \in L^1(\mathbb{R}^n)$, then

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f(x) \quad \text{a.e}$$

This is of course an extension of the fundamental Theorem of Calculus, and is called the Lebesgue differentiation theorem.

The main ingredient in the proof of this theorem is the analysis of the Hardy-Littlewood maximal function..

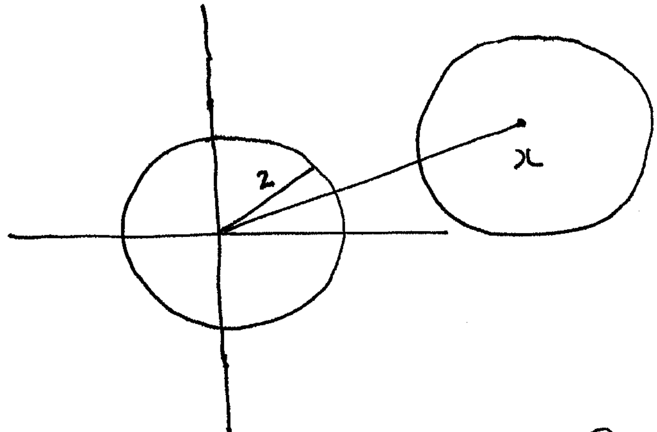
Let $f \in L^1(\mathbb{R}^n)$ and define.

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

Proposition 1: $M(f) \notin L^1(\mathbb{R}^n)$. In fact, if $f(x) = 0$ for $|x| \geq 2$, then for $|x|$ large,

$$\frac{c_1}{|x|^n} \|f\|_{L^1} \leq M(f)(x) \leq \frac{c_2}{|x|^n} \|f\|_{L^1}.$$

Proof: Let $x \in \mathbb{R}^n$; $|x| > 10$.



Since $f(y) = 0$ if $|y| > 2$, $\int_{B(x,r)} |f(y)|\,dy = 0$ unless $B(x,r) \cap B(0,2) \neq \phi$, then $\exists\, y$ such that $|y - x| \leq r$ and $|y| \leq 2$. Since $|y - x| \geq |x| - |y|$; we have $|x| \leq r + 2 \leq r + \frac{1}{5}|x|$ and hence $|x| \leq 5/4\, r$. The conclusion is that $\int_{B(x,r)} |f(y)|\,dy = 0$ unless $|x| \leq 5/4\, r$.

In this case $|B(x,r)| = C r^n \geq C_1 |x|^n$.

and thus

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy \leq \frac{C}{|x|^n} \int_{B(0,2)} |f|\,dy = \frac{C \|f\|_1}{|x|^n}.$$

So $Mf(x) \leq \dfrac{C}{|x|^n} \|f\|_1$ if $|x| > 10$

Similarly, since $B(x, 2|x|) \supset B(0,2)$ ③

$$M f(x) \geq \frac{1}{B(x, 2|x|)} \int_{B(x,2|x|)} |f(y)|\,dy \geq \frac{c}{|x|^n} \|f\|_{L^1}.$$

$\square$

We know that if $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$

$$\mu(\{x : |f(x)| > \lambda\}) = \frac{1}{\lambda} \int_{\{x : |f(x)| > \lambda\}} \lambda \, dx =$$

$$< \frac{1}{\lambda} \int_{\{x : |f(x)| > \lambda\}} |f(x)|\,dx \leq \frac{1}{\lambda} \|f\|_{L^1}.$$

<u>Definition:</u> We say that f is weak $L^1(\mathbb{R}^n)$ if for all $\lambda > 0$

$$\lambda\, \mu(\{x : |f(x)| > \lambda\}) \leq C$$

<u>Example</u>: $f(x) = |x|^{-n}$ if $|x| \geq 1$, $f(x) = 0$ if $|x| < 1$

$$\{x : f(x) > \lambda\} = \{x : |x| < 1/\lambda^{1/n}\}$$

and so $\mu(\{x : f(x) > \lambda\}) \leq C/\lambda$

But $f \notin L^1(\mathbb{R}^n)$

Theorem (Hardy–Littlewood) Let $f \in L^1(\mathbb{R}^n)$, then for any $\lambda > 0$

$$\mu(\{x : M(f)(x) > \lambda\}) \le \frac{C(n)}{\lambda} \|f\|_{L^1}$$

Proof: For $\lambda > 0$, let

$$\mathcal{O}_\lambda = \{x : Mf(x) > \lambda\}$$

If $\mathcal{O}_\lambda = \phi$ we are done. Otherwise, for each $x \in \mathcal{O}_\lambda$ there exists r_x such that

$$\frac{1}{|B(x, r_x)|} \int_{B(x,r_x)} |f(y)|\, dy > \lambda$$

Therefore $\mathcal{O}_\lambda \subset \bigcup_{x \in \mathcal{O}_\lambda} B(x, r_x)$, $B(x, r_x)$ open ball

We know that

$$\mu(\mathcal{O}_\lambda) = \sup\{\mu(K) : K \subset \mathcal{O}_\lambda, K \text{ compact}\}$$

So if we prove that for every $K \subset \mathcal{O}_\lambda$ compact, $K \subset \mathcal{O}_\lambda$ one has

$$\mu(K) \le \frac{C(n)}{\lambda} \|f\|_{1},$$ where

$C(n)$ does not depend on K, we are done.

For each $K \subset \mathcal{O}_\lambda$ we also have

$$K \subset \bigcup_{x \in \mathcal{O}_\lambda} B(x, r_x)$$

But, since K is compact, there exists a finite subcover $B(x_j, r_{x_j})$, $j = 1, \dots, m$, such that

$$K \subset \bigcup_{j=1}^{m} B(x_j, r_j) \qquad r_j = r_{x_j}$$

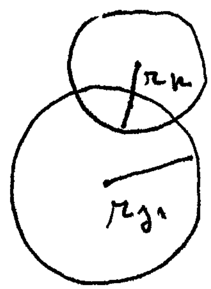
<u>Lemma:</u> (Vitali's covering lemma) Let $B_1, \dots, B_m$, be a family of open balls in $\mathbb{R}^n$, then there exists a subfamily $B_{j_1}, \dots, B_{j_k}$ <u>which are</u> <u>disjoint</u> and satisfy

$$\bigcup_{j=1}^{m} B_j \subset \bigcup_{l=1}^{k} 3 B_{j_l}$$

<u>Proof:</u> <u>Step 1</u> Find ~~Pick~~ the largest of the radii of the balls and pick one ball ~~which~~ with this radius.

If there are more than one, just pick one of them. Let B_{j1} denote this chosen ball.

Step 2 If there is a ball B_k such that $B_k \cap B_{j1} \neq \phi$, then $B_k \subset 3 B_{j1}$.



If $y \in B_k$, and p_k is the center of B_k
$|y - p_k| < r_k$

Let p_{j1} be the center of B_{j1}. then

$$|y - p_{j1}| \leq |y - p_k| + |p_k - p_{j1}| \leq r_k + 2r_{j1} < 3r_{j1}$$

So $y \in 3 B_{j1}$.

Step 3 Now repeat the argument for the collection $\{B_1, \ldots, B_m\} \setminus \{B_{j1}\}$ ///

Therefore, since $K \subset \bigcup_{j=1}^{m} B(x_j, r_j)$, there exist $B(x_{jl}, r_{jl})$; $l = 1, \ldots, k$, such that

$$B(x_{jl}, r_{jl}) \cap B(x_{jk}, r_{jk}) = \phi$$

and
$$K \subset \bigcup_{\ell=1}^{k} 3\,B(x_{j\ell}, r_{j\ell})$$

In particular this implies that

$$\mu(K) \leq \sum_{\ell=1}^{k} |3\,B(x_{j\ell}, r_{j\ell})| \leq 3^n \sum_{\ell=1}^{k} |B(x_{j\ell}, r_{j\ell})|$$

But for each of these balls

$$\frac{1}{|B(x_{j\ell}, r_{j\ell})|} \int_{B(x_{j\ell}, r_{j\ell})} |f(y)|\,dy > \lambda$$

therefore

$$|B(x_{j\ell}, r_{j\ell})| \leq \frac{1}{\lambda} \int_{B(x_{j\ell}, r_{j\ell})} |f(y)|\,dy$$

So we conclude that

$$\mu(K) \leq \frac{3^n}{\lambda} \sum_{\ell=1}^{k} \int_{B(x_{j\ell}, r_{j\ell})} |f(y)|\,dy \leq \frac{3^n}{\lambda} \|f\|_{L^1}$$

since the balls are disjoint. $\square$