Vitali's Covering Lemma

Definition: A family $\mathcal{V}$ of closed intervals of $\mathbb{R}$ is said to be a Vitali covering of a subset $E \subset \mathbb{R}^1$ if for any $x \in E$ and any $\varepsilon > 0$, there exists an interval $I \in \mathcal{V}$ such that $x \in I$ and $|I| < \varepsilon$

Theorem: Let $E \subset \mathbb{R}$ with $|E|_e < \infty$ and let $\mathcal{V}$ be a Vitali covering of E. Then there exists a countable family $\{I_k\}$, $I_k \in \mathcal{V}$, $I_j \cap I_k = \phi$ if $j \neq k$, such that

$$\left| E \setminus \bigcup_{k=1}^{\infty} I_k \right| = 0$$

Proof: **Step 1** Since $|E|_e < \infty$, there exists an open set $\mathcal{O} \subset \mathbb{R}$ such that $E \subset \mathcal{O}$ and $|\mathcal{O}| < \infty$. Discard the elements of $\mathcal{V}$ which are not contained in $\mathcal{O}$. Let $\mathcal{V}_1$ denote $\{I \in \mathcal{V} ; I \subset \mathcal{O}\}$. Again $\mathcal{V}_1$ is a Vitali covering of E.

Step 2 Pick an interval $I_1 \in \mathcal{V}_1$. If

$|E \setminus I_1|_e = 0$ we are done. If not, let

$$\mathcal{O}_1 = \mathcal{O} \setminus I_1 .$$

$\mathcal{O}_1$ is open and not empty. Let

$$\mathcal{V}_2 = \{ I \in \mathcal{V}_1 : I \subset \mathcal{O}_1 \}$$ and let

$$k_1 = \sup \{ |I| : I \in \mathcal{V}_2 \} > 0 .$$

Pick an interval $I_2 \in \mathcal{V}_2$ with $|I_2| > k_1/2$.

It is clear that $I_1 \cap I_2 = \phi$.

If $|E \setminus (I_1 \cup I_2)|_e = 0$ we are done. If not,

Let

$$\mathcal{O}_2 = \mathcal{O} \setminus (I_1 \cup I_2)$$

$\mathcal{O}_2$ is open and not empty. Let

$$\mathcal{V}_3 = \{ I \in \mathcal{V}_2 : I \subset \mathcal{O}_2 \}$$

and

$$k_2 = \sup \{ |I| : I \in \mathcal{V}_3 \}$$

Pick $I_3 \in \mathcal{V}_3$ with $|I_3| > k_2/2$.

If this selection stops after a finite number of steps we are done. If not, we have constructed a family of intervals $\{I_k\}$ with $I_j \cap I_k = \phi$ such that

$$\bigcup_{j=1}^{\infty} I_j \subset \mathcal{O} \quad \text{and} \quad \sum_{j=1}^{\infty} |I_j| \leq |\mathcal{O}| < \infty$$

In particular this says that that

$$(*) \quad k_N = \sup\left\{ I : I \subset \mathcal{O} \setminus \bigcup_{j=1}^{N} I_j \right\} \to 0 \text{ as } N \to \infty$$

and if $\delta > 0$, there exists N. such that

$$\sum_{k=N+1}^{\infty} |I_k| < \delta.$$

Step 3: $\quad E \setminus \left(\bigcup_{k=1}^{N} I_k \right) \subset \bigcup_{k=N+1}^{\infty} 5 I_k$

By assumption $E \setminus \bigcup_{k=1}^{\infty} I_k \neq \phi$, then for every $x \in E \setminus \bigcup_{k=1}^{\infty} I_k$, there exists $I \in \mathcal{V}$, $I \subset \mathcal{O} \setminus \bigcup_{k=1}^{\infty} I_k$ such that $x \in I$. But $(*)$ implies that there exists some $m \geq N+1$ such that

$$I \cap I_m \neq \phi$$

If not we would have $|I| \le k_N$ for all N.

But this is impossible since $k_N \to 0$ as $N \to \infty$.

Let m_0 be the smallest m such that $I \cap I_m \neq \phi$.

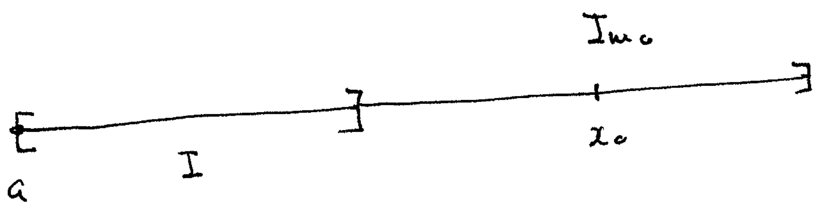
In this case $I \cap \bigcup_{k=1}^{m_0-1} I_k = \phi$ and by the

Construction

$$|I| \le k_{m_0-1} \le 2\,|I_{m_0}|$$

Since $I \cap I_{m_0} \neq \phi$, it follows that $I \subset 5\, I_{m_0}$

If $a \in I$ and x_0 is the center of I_{m_0}, then



$$d(a, x_0) \le |I| + \frac{1}{2}|I_{m_0}| \le 2\,|I_{m_0}| + \frac{1}{2}|I_{m_0}| = \frac{5}{2}|I_{m_0}|$$

$$\Rightarrow \quad a \in 5\, I_{m_0}.$$

Step 4: $$\left| E \setminus \bigcup_{k=1}^{N} I_k \right| \le \sum_{k=1}^{\infty} |5\, I_k| \le 5 \sum_{k=1}^{\infty} |I_k|$$

$$\le 5\delta.$$

$\square$.

Monotone Functions:

Theorem: If $f: (a,b) \longrightarrow \mathbb{R}$ is monotone, $-\infty \le a < b \le \infty$, then the set of discontinuities of f is at most countable.

Proof. Suppose that f is non-decreasing, i.e if $y > x$ then $f(y) \ge f(x)$. Then for every $x \in (a,b)$ the lateral limits exist.

$$f(x^+) = \lim_{\substack{t \to x \\ t > x}} f(t) \quad ; \quad f(x^-) = \lim_{\substack{t \to x \\ t < x}} f(t)$$

and $f(x^-) \le f(x) \le f(x^+)$

Moreover if $x < y$; $f(x^+) \le f(y^-)$.

To see that, let

$$A = \sup \{ f(t) : a < t < x \} \quad , \quad A \le f(x)$$

and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $a < x - \delta < x$ and $A - \varepsilon < f(x-\delta) \le A$,

thus $A = \lim_{t \to x^-} f(t) = f(x^-)$.

Similarly $B = \inf \{ f(t), t > x \} = \lim_{t \to x^+} f(t)$.

If $a < x < y < b$

$$f(x^+) = \inf_{b > t > x} f(t) \leq \inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t) = f(y^-).$$

Let $D = \{ x \in (a,b) : f(x^-) < f(x^+) \}$ = set of discontinuities of f.

For each $x \in D$, pick a rational $r(x)$ with

$$f(x^-) < r(x) < f(x^+)$$

But if $x_1, x_2 \in D$; $x_1 < x_2$, we have

$$r(x_1) < f(x_1^+) \leq f(x_2^-) < r(x_2)$$

So the function $r : D \longrightarrow \mathbb{Q}$ is injective

and so D is at most countable.

<u>Theorem</u>: If $f : (a,b) \longrightarrow \mathbb{R}$, $-\infty \leq a < b \leq \infty$, is a monotone function, then $f'(x)$ exists almost everywhere.

<u>Remark</u>: It is not true that the set of points where an arbitrary monotone function is differentiable is countable.

Let $E \subset (a,b)$ be a set of measure zero. For each $k \in \mathbb{N}$, pick an open set $G_k \subset (a,b)$ such that

$$E \subset G_k \quad \text{and} \quad |G_k| < 1/2^k .$$

Let

$$f_k(x) = |G_k \cap (a,x)|$$

f_k is continuous, increasing, and $f_k(x) \le 1/2^k$ for $x \in (a,b)$. Let

$$f(x) = \sum_{k=1}^{\infty} f_k(x)$$

the series converges uniformly and so f is continuous on (a,b). f is also increasing.

On the other hand if $x \in E$ and h is small enough,

$$[x, x+h] \subset G_k$$

therefore

$$f_k(x+h) - f_k(x) = |G_k \cap (a, x+h)| - |G_k(a, x)|$$

$$= |(x, x+h)| = h$$

~~therefore~~ and so $\dfrac{f_k(x+h) - f_k(x)}{h} = 1$.

Thus, for every N, there exists $h_0 > 0$ so that

if $0 < h < h_0$

$$\frac{f(x+h) - f(x)}{h} \geq \sum_{k=1}^{N} \frac{f(x+h) - f(x)}{h} \geq N.$$

thus $\displaystyle\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \infty$ for all $x \in E$.