Theorem: Let $f : (a,b) \longrightarrow \mathbb{R}$, $-\infty \leq a < b \leq \infty$, be a monotone function. Then $f'(x)$ exists for almost every $x \in (a,b)$.

Proof: Suppose f is non-decreasing, i.e. $f(x) \geq f(y)$ if $x > y$. Our goal is to prove that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ exists for almost every $x \in (a,b)$.

So we define, for $x \in (a,b)$ and h such that $x+h \in (a,b)$,

$$Df(x,h) = \frac{f(x+h) - f(x)}{h},$$

and let

$$D^+ f(x) = \limsup_{h \to 0^+} Df(x,h) \,; \quad D_+ f(x) = \liminf_{h \to 0^+} Df(x,h)$$

$$D^- f(x) = \limsup_{h \to 0^-} Df(x,h) \,; \quad D_- f(x) = \liminf_{h \to 0^-} Df(x,h)$$

$f'(x)$ exists if and only if

$$(1) \quad D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x)$$

It is obvious that

$$0 \le D_- f(x) \le D^- f(x) \quad \text{and} \quad 0 \le D_+ f(x) \le D^+ f(x)$$

for every $x \in (a,b)$. We will prove that

$$D^- f(x) \le D_+ f(x) \quad \text{and} \quad D^+ f(x) \le D_- f(x)$$

for almost every $x \in (a,b)$.

The main point of the proof is the following

Lemma: The sets

$$B_1 = \{x \in (a,b) : D^- f(x) > D_+ f(x)\} \quad \text{and}$$

$$B_2 = \{x \in (a,b) : D^+ f(x) > D_- f(x)\}$$

have measure zero.

Proof: We will prove that $|B_2| = 0$. The proof that $|B_1| = 0$ is identical and will be left as exercise.

For $u, v \in \mathbb{Q}$, $u > v > 0$, let

$$B_{u,v} = \{x \in (a,b) : D^+ f(x) > u > v > D_- f(x)\}.$$

We claim that $B_2 = \bigcup_{u,v \in \mathbb{Q}} B_{u,v}$ and the result follows if we show that $|B_{u,v}| = 0$.

Let $\eta = |B_{u,v}|_e$. By restricting the domain of f, we may assume that $a > -\infty$ and $b < \infty$. The general case follows from this one. Since $B_{u,v} \subset (a,b)$, then $\eta < \infty$. In this case, for any $\varepsilon > 0$ $\exists$ an open subset $\mathcal{O} \subset (a,b)$ such that

$$B_{u,v} \subset \mathcal{O} \quad \text{and} \quad |\mathcal{O}| \leq |B_{u,v}| + \varepsilon = \eta + \varepsilon$$

For each $x \in B_{u,v}$, we have by definition

$$v > D_- f(x) = \liminf_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

therefore there exists a sequence $h_{x,n} \to 0$, $h_{x,n} > 0$, such that for every n

$$J_n(x) = [x - h_{x,n}\,;\, x] \in \mathcal{O} \quad \text{and} \quad \frac{f(x - h_{x,n}) - f(x)}{-h_{x,n}} < v$$

therefore on the intervals $J_n(x)$ we have

$$f(x) - f(x - h_{x,n}) < v\, h_{x,n}$$

The intervals $J_n(x)$ form a Vitali covering of $B_{u,v}$, and therefore there exists a finite collection

$I_1 = [x_1 - h_1, x_1], \ldots, I_N = [x_N - h_N, x_N]$

such that $I_j \cap I_k = \phi$ if $j \neq k$ and

$$\left| B_{u,v} \setminus \bigcup_{j=1}^{N} I_j \right| < \varepsilon, \quad \text{with } \varepsilon \text{ chosen above.}$$

On the other hand, for each $x \in B_{u,v} \cap \bigcup_{j=1}^{N} \mathring{I}_j$

$$D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h} > u,$$

so there is a sequence $k_{x,m} \to 0$, $k_{x,m} > 0$, such that

that $J_m(x) = [x, x + k_{x,m}] \subset \bigcup_{j=1}^{N} \mathring{I}_j$ and that

$$f(x + k_{x,m}) - f(x) > u\, k_{x,m} \quad \text{for all } m$$

The family $J_m(x)$, $m \in \mathbb{N}$, $x \in B_{u,v} \cap \bigcup_{j=1}^{N} \mathring{I}_j$, form a

Vitali covering of $\tilde{B}_{uv} = B_{uv} \cap \bigcup_{j=1}^{N} \mathring{I}_j$. Therefore

there exists a finite collection

$J_1 = [x_1', x_1' + k_1], \ldots, J_M = [x_M', x_M' + k_M]$

with $J_j \cap J_k = \phi$

such that $\left| \tilde{B}_{uv} \setminus \bigcup_{i=1}^{M} J_i \right|_e < \varepsilon$.

Let

$$\Delta J = \sum_{i=1}^{M} \left(f(x_i' + k_i) - f(x_i') \right) \geq \sum_{i=1}^{M} u\, k_i$$

$$\Delta I = \sum_{j=1}^{N} \left(f(x_j) - f(x_j - h_j) \right) < \sum_{j=1}^{N} v\, h_j$$

Since $[x_i' ; x_i' + k_i] \subset \bigcup_{j=1}^{N} \mathring{I}_j$, $\mathring{I}_j \cap \mathring{I}_k = \phi$,

$[x_i' ; x_i' + k_i] \subset \mathring{I}_\ell$ for some $\ell \in \{1, \dots, N\}$. Therefore,

since f is non-decreasing $\Delta J \leq \Delta I$.

So we obtain

$$u \sum_{i=1}^{M} k_i = u \sum_{i=1}^{M} |J_i| < \Delta J < \Delta I < v \sum_{j=1}^{N} h_j$$

$$= v \sum_{j=1}^{N} |I_j| \leq v |\mathcal{O}|$$

$$\leq v(\eta + \varepsilon).$$

But we can write

$$B_{uv} = \left(B_{uv} \setminus \bigcup_{j=1}^{N} \mathring{I}_j \right) \cup \left(\underbrace{B_{uv} \cap \bigcup_{j=1}^{N} \mathring{I}_j}_{= \tilde{B}_{uv}} \right)$$

$$= \left(B_{uv} \setminus \bigcup_{j=1}^{N} \mathring{I}_j \right) \cup \left(\tilde{B}_{uv} \setminus \bigcup_{j=1}^{M} J_j \right) \cup \bigcup_{j=1}^{M} J_j$$

Therefore

$$\eta = |B_{uv}|_e \leq 2\varepsilon + \sum_{j=1}^{M} |J_j|$$

and so : $\displaystyle\sum_{j=1}^{M} |J_j| \geq \eta - 2\varepsilon$

Since $\displaystyle u \sum_{j=1}^{M} |J_j| \leq \Delta J \leq v(\eta + \varepsilon)$

we obtain

$$u(\eta - 2\varepsilon) \leq v(\eta + \varepsilon) \quad \text{for all } \varepsilon > 0$$

Therefore $u\eta \leq v\eta$, $(u - v)\eta \leq 0$

. So $\eta = 0$. $\square$.

<u>Theorem:</u> If $f:(a,b)\to\mathbb{R}$ is non-decreasing and $-\infty < a < b < \infty$, then

$$\int_a^b f'(x)\,dx \le f(b^-) - f(a^+)$$

<u>Proof:</u> Extend f to $\mathbb{R}$ by setting $f(x) = f(a^+)$ if $x \le a$ and $f(x) = f(b^-)$ if $x \ge b$

We know that

$$f'(x) = \lim_{n\to\infty} \frac{f(x+1/n) - f(x)}{1/n} \quad \text{a.e in } (a,b)$$

Let $f_n(x) = n\left(f(x+1/n) - f(x)\right)$. Then, by Fatou's lemma:

$$\int_a^b f'(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx \le \liminf_{n\to\infty} \int f_n(x)\,dx$$

But

$$\int_a^b f_n(x)\,dx = n\left[\int_a^b \left(f(x+1/n) - f(x)\right)dx\right] =$$

$$= n\left[\int_{a+1/n}^{b+1/n} f(x)\,dx - \int_a^b f(x)\,dx\right] =$$

$$= n \int_b^{b+1/n} f(x)\,dx - n \int_a^{a+1/n} f(x)\,dx$$

But for $x \geq b$, $f(x) = f(b^-)$ and for $x \geq a$,

Since f is non-decreasing $f(x) \geq f(a^+)$

Therefore $\int_a^b f_n(x)\,dx \leq f(b^-) - f(a^+)$ $\square$.