Functions of Bounded Variation.

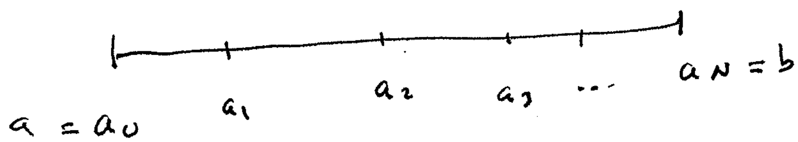
<u>Definition</u>: $f : [a,b] \longrightarrow \mathbb{R}$ is of bounded variation if

$$V(f, a, b) = \sup_{\mathcal{P}} \sum_{j=1}^{N} | f(a_j) - f(a_{j-1}) | < \infty$$

$\mathcal{P}$ is a partition of $[a,b]$. We denote this class by $BV([a,b])$

<u>Examples</u>:

1) If f is monotone, then $f \in BV([a,b])$.



Suppose f is non-decreasing

$$\sum_{j=1}^{N} |f(a_j) - f(a_{j-1})| = \sum_{j=1}^{N} f(a_j) - f(a_{j-1}) = f(b) - f(a).$$

2) Lipschitz functions are of bounded variation

$$|f(y) - f(x)| \leq C|x - y|.$$

$$V(f, a, b) \leq C|b - a|.$$

3) Continuous functions are not necessarily of Bounded Variation.

Let $f(x) = x \sin\left(\frac{1}{x}\right)$; $x \in [0, \frac{2}{\pi}]$ and

Consider the partition $P_N = \left\{0, \frac{2}{N\pi} ; \frac{2}{(N-2)\pi}, \dots, \frac{2}{\pi}\right\}$

N odd. So, if k is odd

$$\left| f\left(\frac{2}{k\pi}\right) - f\left(\frac{2}{(k-2)\pi}\right) \right| = \left| \frac{2}{k\pi} \sin\left(\frac{k\pi}{2}\right) - \frac{2}{(k-2)\pi} \sin\left((k-2)\frac{\pi}{2}\right) \right|$$

$$> \frac{2}{k\pi}$$

$$\sum_{k=1}^{K} \left| f\left(\frac{2}{k\pi}\right) - f\left(\frac{2}{(k-2)\pi}\right) \right| \geq \sum_{k=1}^{K} \frac{2}{k\pi} \geq \frac{2}{\pi} \log K.$$

If $[c, d] \subset [a, b]$ we have.

$$V(f, c, d) \leq V(f, a, b)$$ and for $x \in [a, b]$

define

$$V(x) = f(f, a, x)$$

So V has the following properties.

(1) $V(0) = 0$

(2) $V(x)$ is non-decreasing

(3) $|f(x+h) - f(x)| \leq V(f, x, x+h)$ $\quad a \leq x \leq x+h \leq b.$

Proposition: $V(x) + V(f, x; x+h) = V(x+h)$

if $a \le x < x+h \le b$.

Proof: $V(x) = \sup\limits_{\mathcal{P}([a,x])} \sum |f(a_j) - f(a_{j-1})|$

$$V(f; x, x+h) = \sup_{\mathcal{P}([x, x+h])} \sum |f(a_j) - f(a_{j-1})|.$$

For any two partitions $\mathcal{P}_1$ of $[a, x]$ and $\mathcal{P}_2$ of $[x, x+h]$

$\mathcal{P}_1 \cup \mathcal{P}_2$ give a partition of $[a, x+h]$ so.

$$\sum_{\mathcal{P}_1 \cup \mathcal{P}_2} |f(a_j) - f(a_{j-1})| \le V(x+h)$$

So $V(x) + V(f; x, x+h) \le V(x+h)$

On the other hand if $\mathcal{P}_1, \mathcal{P}_2$ are partitions of

$[a, x+h]$ and $\mathcal{P}_1 \subset \mathcal{P}_2$

$$\sum_{\mathcal{P}_1} |f(a_j) - f(a_{j-1})| \le \sum_{\mathcal{P}_2} |f(a_j) - f(a_{j-1})|$$

So if $\mathcal{P}$ is a partition of $[a, x+h]$

$\mathcal{P} \subset \mathcal{P} \cup \{x\} = \mathcal{P}_1 = \mathcal{P}([a, x]) \cup \mathcal{P}([x, x+h])$

$$\sum_{\mathcal{P}} |f(a_j) - f(a_{j-1})| \le \sum_{\mathcal{P}([a,x])} |f(a_j) - f(a_{j-1})| + \sum_{\mathcal{P}([x,x+h])} |f(a_j) - f(a_{j-1})|$$

$$\le V(x) + V(f, x, x+h).$$

Proposition: Let $f \in BV([a,b])$. Then $V+f$ and $V-f$ are non-decreasing functions.

Proof: If $a \leq x < y < b$, we have

$$(V+f)(y) - (V+f)(x) = V(y) - V(x) + f(y) - f(x)$$

$$= V(f, x, y) + (f(y) - f(x))$$

Since $V(f, x, y) \geq |f(y) - f(x)|$

$$(V+f)(y) \geq (V+f)(x).$$

Similarly

$$(V-f)(y) - (V-f)(x) = V(y) - V(x) + f(x) - f(y)$$

$$\geq |f(y) - f(x)| + (f(x) - f(y)) \geq 0$$

Corollary: If $f \in BV([a,b])$, then there exist non-decreasing functions F_1 and F_2 such that

$$f = F_1 - F_2$$

Theorem If $f: (a,b) \longrightarrow \mathbb{R}$ is non-decreasing, $-\infty < a < b < \infty$, then f' is measurable and

$$\int_a^b f'(x)\,dx \leq f(b^-) - f(a^+)$$

So if f is bounded, $f' \in L^1((a,b))$.

Proof: Extend f to $\mathbb{R}$ by setting $f(x) = f(a^+)$ if $x \leq a$ and $f(x) = f(b^-)$ if $x \geq b$. Let

$$f_n(x) = \frac{f(x + 1/n) - f(x)}{1/n} \qquad x \in \mathbb{R}, \quad n = 1, 2, \dots$$

Since $\lim_{n\to\infty} f_n(x) = f'(x)$ a.e then $f'(x)$ is measurable.

Fatou's lemma implies that

$$\int_a^b f'(x)\,dx \leq \liminf_{n\to\infty} \int_a^b f_n(x)\,dx =$$

$$\liminf_{n\to\infty} n\left[\int_a^b f(x+1/n)\,dx - \int_a^b f(x)\,dx\right]$$

$$= \liminf_{n\to\infty} \left[\int_b^{b+1/n} f(x)\,dx - \int_a^{a+1/n} f\,dx\right] \leq f(b^-) - f(a^-).$$

<u>Corollary</u>: Suppose $f \in BV([a,b])$ on a bounded interval $[a,b]$. Then $f' \in L^1([a,b])$ and

$$\int_I |f'(x)|\,dx \leq V(f,a,b)$$

<u>Proof</u>: Since $f(x) = f_1(x) - f_2(x)$, with $f_i(x)$ non-decreasing, $f'(x)$ exists a.e. We also know that for $x \in (a,b)$ and $x+h \in (a,b)$

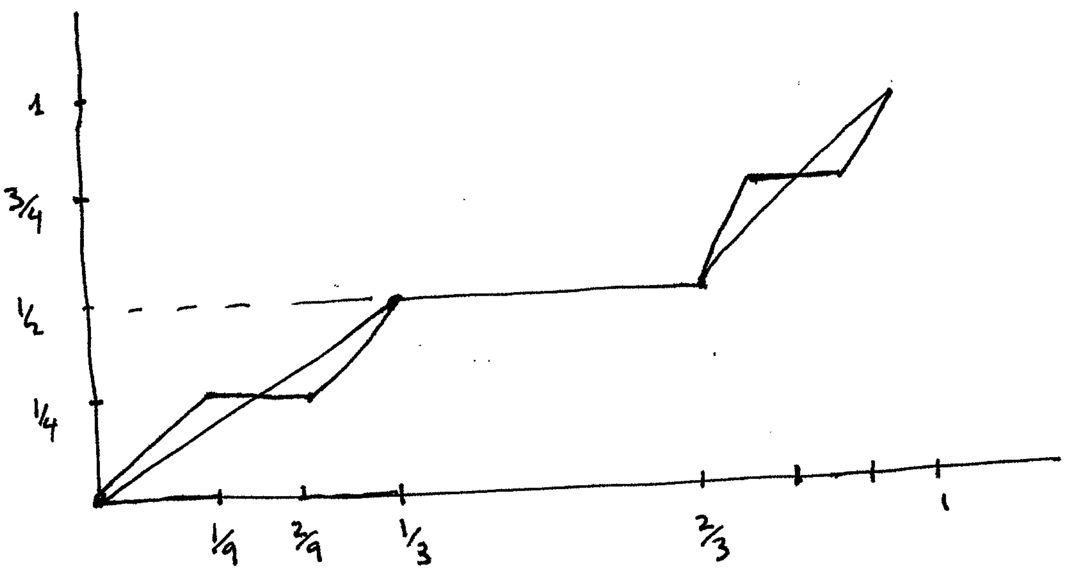
$$|f(x+h) - f(x)| \leq V(x+h) - V(x)$$

therefore $|f'(x)| \leq V'(x)$ and the inequality follows from the previous theorem.

The Cantor function:

Let $C_0 = [0,1]$ and $C_1 = [0, 1/3] \cup [2/3, 1] = I_1^1 \cup I_1^2$

$$f_1(x) = \begin{cases} \frac{3}{2}x ; & 0 \le x \le 1/3 \\ 1/2 , & 1/3 \le x \le 2/3 \\ \frac{3}{2}x - 1/2 , & 2/3 \le x \le 1 \end{cases}$$



Now repeat this construction on I_1^1 and I_2^1

$$C_2 = I_2^1 \cup I_2^2 \cup I_2^3 \cup I_2^4$$

Let $f_2^1(x) = \frac{1}{2} f_1(3x)$ if $0 \le x \le 1/3$

$f_2^2(x) = f_2^1(x - 2/3) + 1/2$ if $2/3 \le x \le 1$

Repeat this construction at every step. This gives a sequence of functions $f_n(x)$ which satisfy

1) f_n is ~~...~~ non-decreasing and continuous

2) $|f_n(x) - f_{n+1}(x)| \leq \frac{1}{2^n}$ $\quad x \in [0,1]$

Since $f_m(x) - f_k(x) = \sum_{j=k}^{m-1} f_{j+1}(x) - f_j(x)$

We find that $|f_m(x) - f_k(x)| \leq \sum_{j=k}^{\infty} 2^{-j} \leq 2^{-k+1}$, $x \in [0,1]$

So the sequence converges uniformly. Let

$$f(x) = \lim_{n \to \infty} f_n(x).$$

then $f(x)$ is continuous and non-decreasing and $f(x)$ is constant on the complement of the Cantor set.

So $f'(x) = 0$ a.e.

In this case

$$\int_0^1 f'(x)\,dx = 0 \neq f(1) - f(0) = 1.$$