Signed Measures. Let $(X, \mathcal{S})$ be a measurable space. A map

$$\nu : \mathcal{S} \longrightarrow [-\infty, \infty) \quad \text{or} \quad \nu : \mathcal{S} \longrightarrow (-\infty, \infty]$$

is a signed measure if.

(1) $\nu(\phi) = 0$

(2) $E_j \in \mathcal{S}$, $j = 1, 2, \dots$, $E_j \cap E_k = \phi$, then

$$\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu(E_j)$$

Example: Let $\mu_1 : \mathcal{S} \longrightarrow [0, \infty]$; $\mu_2 : \mathcal{S} \rightarrow [0, \infty)$ be positive measures. Then

$$\nu = \mu_1 - \mu_2 \quad \text{and} \quad \tilde{\nu} = \mu_2 - \mu_1$$

are signed measures.

Def. If ν is a signed measure on $(X, \mathcal{S})$, a set $E \in \mathcal{S}$ is positive (negative) if $\nu(E \cap A) \geq 0$ (or ≤ 0) for all $A \in \mathcal{S}$.

Theorem: If μ is a signed measure on $(X, \mathcal{T})$, then there exist two measurable sets $A, B \in \mathcal{T}$ such that

(1) $A \cap B = \phi$ (2) $A \cup B = X$

(3) A is positive and B is negative with respect to μ.

We say that A and B form a Hahn decomposition of X with respect to μ.

Remarks:

(1) If $X = A_1 \cup B_1$ and $X = A_2 \cup B_2$ are two Hahn decompositions of a signed measure μ, then for any $E \in \mathcal{T}$

$$\mu(E \cap A_1) = \mu(E \cap A_2) = \mu(E \cap A_1 \cap A_2)$$

$$\mu(E \cap B_1) = \mu(E \cap B_2) = \mu(E \cap B_1 \cap B_2)$$

To see that $E \cap A_1 = (E \cap A_1 \cap A_2) \cup (E \cap A_1 \cap B_2)$

Since A_1 is positive $\mu(E \cap A_1 \cap B_2) \geq 0$, but B_2 is negative so $\mu(E \cap A_1 \cap B_2) \leq 0$.

Hence $\mu(E \cap A_1 \cap B_2) = 0$.

(2) A Hahn decomposition is not unique.

Let $(X, \mathcal{F}, \mu)$ be a measure space, μ a positive measure and $f \in L^1(X, d\mu)$, not necessarily positive. Let

$$\nu(E) = \int_E f \, d\mu .$$

ν is a signed measure. Let

$A_1 = \{x : f(x) > 0\}$; $B_1 = \{x : f(x) \leq 0\}$

$A_2 = \{x : f(x) \geq 0\}$; $B_2 = \{x : f(x) < 0\}$

$X = A_1 \cup B_1$; $X = A_2 \cup B_2$.

ν is positive on A_j and negative on B_j, $j = 1, 2$.

We need the following

<u>Lemma</u>: Let μ be a signed measure on $(X, \mathcal{F})$. If $E \subset F$, $E, F \in \mathcal{F}$ and $|\mu(F)| < \infty$, then $|\mu(E)| < \infty$

Proof: the point is that μ does not assume $+\infty$ and $-\infty$, it only assumes either one of them.

Since
$$\mu(F) = \mu(E) + \mu(F \setminus E)$$
and $|\mu(F)| < \infty$, then both $|\mu(E)| < \infty$ and $|\mu(F \setminus E)| < \infty$.

Lemma. If $E_j \in \mathcal{T}$, $j \in \mathbb{N}$, are positive (or negative) sets, then $E = \bigcup_{j=1}^{\infty} E_j$ is positive (or negative)

Proof: Notice that if E_1 and E_2 are positive sets, then $E_1 \cap E_2$, $E_1 \cap E_2^c$ and $E_2 \cap E_1^c$ are positive sets. Since
$$E_1 \cup E_2 = (E_1 \cap E_2) \cup (E_1 \cap E_2^c) \cup (E_2 \cap E_1^c)$$
and these sets are disjoint, for any set $A \in \mathcal{T}$; $\mu(A \cap (E_1 \cup E_2)) \geq 0$. So $E_1 \cup E_2$ is positive.

Then, by induction, we deduce that if E_j is positive, $j = 1, 2, \ldots, N$, then

$$\bigcup_{j=1}^{N} E_j = E_N \text{ is positive.}$$

It is also easy to check that if E_j; $j = 1, 2, \ldots,$ are positive and $E_j \cap E_k = \phi$, then $\bigcup_{j=1}^{\infty} E_j$ is positive. Indeed, if $A \in \mathcal{T}$

$$\mu\left(A \cap \bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{j=1}^{\infty} (A \cap E_j)\right) = \sum_{j=1}^{\infty} \mu(A \cap E_j) \geq 0.$$

In general, if $E_j \cap E_k \neq \phi$, we may write.

$$\tilde{E}_1 = E_1; \quad \tilde{E}_2 = E_2 \setminus E_1; \quad \tilde{E}_N = E_N \setminus \left(\bigcup_{j=1}^{N-1} E_j\right)$$

then $\tilde{E}_j \cap \tilde{E}_k = \phi$ and

$$E = \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} \tilde{E}_j.$$

Since $\tilde{E}_j$ is positive E is positive.

Let us suppose that $-\infty < \mu(E) \leq \infty \ \forall E \in \mathcal{S}$.

Let

$$\beta = \inf \mu(B_0) : B_0 \in \mathcal{S} \text{ and } B_0 \text{ is negative.}$$

There is a sequence $B_j \in \mathcal{S}$, $j = 1, 2, \ldots$ such that $\mu(B_j) \to \beta$ and B_j is negative. Then

$$B = \bigcup_{j=1}^{\infty} B_j \text{ is negative and}$$

$$\mu(B) \leq \mu(B_j) \quad \left(\begin{array}{l} \text{since } B = B_j \cup (B \setminus B_j) \\ \mu(B) = \mu(B_j) + \mu(B \cap B_j^c) \end{array} \right)$$

therefore $\beta = \mu(B)$. Since $\mu(E) > -\infty \ \forall E \in \mathcal{S}$,

$\beta > -\infty$.

We will prove that $A = X \setminus B$ is a positive set. Then $X = A \cup B$ with A positive and B negative, and thus will prove the theorem.

Suppose A is not positive. Then there exists a subset $E_0 \subset A$ such that $\mu(E_0) < 0$.

E_0 cannot be a negative set, otherwise $B \cup E_0$ would be negative and

$$\mu(B \cup E_0) = \mu(B) + \mu(E_0) < \mu(B) = \beta$$

which is a contradiction

So $\exists$ $E_1 \subset E_0$ with $\mu(E_1) > 0$.

Let $m_1 \in \mathbb{N}$ be the smallest integer such that $\exists$ $E_1 \subset E_0$ with $\mu(E_1) \geq 1/m_1$

Since $E_1 \subset E_0$, $\mu(E_1) < \infty$. But

$$E_0 = E_1 \cup (E_0 \setminus E_1)$$

and hence $\mu(E_0 \setminus E_1) = \mu(E_0) - \mu(E_1) < 0$

But as above $E_0 \setminus E_1$ cannot be a negative set. So it must contain a subset $E_2 \in \mathcal{J}$, $E_2 \subset E_0 \setminus E_1$ with $\mu(E_2) > 0$. Let m_2 be the smallest integer such that $\exists$ $E_2 \subset E_0 \setminus E_1$ with $\mu(E_2) \geq 1/m_2$.

Continuing this process, we find a sequence $m_k \in \mathbb{N}$, $k = 1, 2, \ldots$, such that m_k is the smallest positive integer such that $\exists\ E_k \subset E_0 \setminus \bigcup_{j=1}^{k-1} E_j$

with

$$\mu(E_k) \geq 1/m_k.$$

Notice that any subset $A \subset E_0 \setminus \bigcup_{j=1}^{k} E_j$

satisfies

$$\mu(A) \leq \frac{1}{m_k - 1}$$

Since $\bigcup_{k=1}^{\infty} E_k \subset E_0$; $\mu(\bigcup E_k) < \infty$ and so $\sum 1/m_k < \infty$. In particular

$$\lim_{k\to\infty} \frac{1}{m_k} = 0$$

Let $F_0 = E_0 \setminus \bigcup_{k=1}^{\infty} E_k$. If $F \subset F_0$, then $\mu(F_0) \leq \frac{1}{m_k - 1}$ $\forall k \Rightarrow \mu(F_0) \leq 0$.

So F_0 is a negative set. But

$$\mu(F_0) = \mu(E_0) - \sum_{k=1}^{\infty} \mu(E_k) < 0.$$

Since $B \cap F_0 = \phi$, $B \cup F_0$ is negative

and $\mu(B \cup F_0) < \mu(B)$. Absurd! $\boxed{//}$.

Set

$$\mu^+(E) = \mu(E \cap A); \quad \mu^-(E) = -\mu(E \cap B)$$

So $\mu(E) = \mu^+(E) - \mu^-(E)$

and at least one of them is finite.