MA 544 10/09/2009

Density of Continuous Functions:

Let $C_0(\mathbb{R}^n) \equiv C_c(\mathbb{R}^n)$ be the space of functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{C}$ (or $\mathbb{R}$) which are continuous and there exists $R > 0$ such that $\varphi(x) = 0$ if $|x| \geq R$. These are called continuous functions of compact support, where

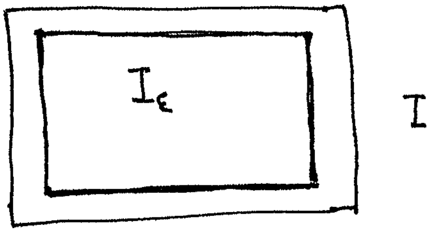
$$\text{Support of } \varphi \equiv \text{Supp } \varphi = \overline{\{x : \varphi(x) \neq 0\}}.$$

Theorem: $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $p \in [1, \infty)$.

Proof:

Step 1: Let $I = [a_1, b_1) \times \cdots \times [a_n, b_n)$. Then $\forall \varepsilon > 0$ there exists $\varphi \in C_c(\mathbb{R}^n)$ such that $0 \leq \varphi \leq \chi_I$ and such that

$$\int (\chi_I(x) - \varphi(x))^p \, d\mu < \varepsilon. \quad p \geq 1$$



Just pick a box $I_\varepsilon \subset I$ such that $\mu(I \setminus I_\varepsilon) < \varepsilon$ and one can take $\varphi(x) = 1$ if $x \in I_\varepsilon$ such $\varphi(x) \le 1$ if $x \in I \setminus I_\varepsilon$; $\varphi(x) = 0$ if $x \notin I$.

<u>Step 2</u>: Let $\mathcal{O} \subset \mathbb{R}^n$ open with $\mu(\mathcal{O}) < \infty$. Then $\forall\ \varepsilon > 0$ there exists $\varphi \in C_c(\mathbb{R}^n)$ such that $0 \le \varphi \le \chi_{\mathcal{O}}$ and

$$\int (\chi_{\mathcal{O}} - \varphi)^p \, d\mu < \varepsilon .$$

<u>Proof</u>: $\mathcal{O} = \bigcup_{j=1}^{\infty} I_j \qquad \mathring{I}_j \cap \mathring{I}_k = \emptyset$

Pick N such that $\sum_{j=N}^{\infty} \mu(I_j) < \varepsilon/2$

For each $j \le N-1$, let $\varphi_j^\varepsilon \in C_c(\mathbb{R}^n)$ be as in step 1 with

$$\int (\chi_{I_j} - \varphi_j^\varepsilon)^p \, d\mu < \varepsilon/2^{j+1}$$

Let $\varphi(x) = \sum_{j=1}^{N-1} \varphi_j^\varepsilon$

Then

$$\int (\chi_{\mathcal{O}} - \varphi)^p \, d\mu \le \sum_{j=1}^{N-1} \varepsilon/2^{j+1} + \sum_{j=N}^{\infty} \mu(J_j)$$

$$< \varepsilon .$$

Step 3: Let $A \subset \mathbb{R}^n$ be Lebesgue measurable and assume that $\mu(A) < \infty$. Then $\forall \varepsilon > 0$ $\exists$ $\varphi \in C_c(\mathbb{R}^n)$ such that

$$\| \chi_A - \varphi \|_p < \varepsilon .$$

Proof: $\exists$ $\mathcal{O}$ open such that $A \subset \mathcal{O}$ and $\mu(\mathcal{O} \setminus A) < \varepsilon/2$. Let φ be such that

$$\| \chi_{\mathcal{O}} - \varphi \|_p < \varepsilon/2$$

Then $\| \chi_A - \varphi \|_p \le \| \chi_A - \chi_{\mathcal{O}} \|_p + \| \chi_{\mathcal{O}} - \varphi \|_p < \varepsilon$

Step 4: Let ~~[illegible]~~ $f = f_+ - f_-$; $f_\pm \geq 0$

Step 5: If $0 \le f$; $f \in L^p$, $p \in [1, \infty)$

Let $S_n = \sum_{j=2}^{n 2^n} \frac{j-1}{2^n} \chi_{f^{-1}([\frac{j-1}{2^n}, \frac{j}{2^n}))}$

Then $s_1 \leq s_2 \leq \cdots \leq s_n \leq \cdots \leq f$ and

$$\lim_{n\to\infty} s_n = f.$$

Since $p \geq 1$ $\|f - s_n\|_p \leq \|f\|_p + \|s_n\|_p \leq 2\|f\|_p$.

the dominated convergence theorem gives that

$$\lim_{n\to\infty} \|f - s_n\|_p = 0.$$

Given $\varepsilon > 0$ pick n s.t $\|f - s_n\|_p < \varepsilon/2$.

Since $f \in L^p$; $\mu\left(f^{-1}\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)\right)\right) < \infty$ $j \geq 2$

~~$\{x : f(x) \neq 0\ f = \emptyset$~~ and therefore by step 3

One can choose $\varphi \in C_c(\mathbb{R}^n)$ such that

$$\|\varphi - s_n\|_{L^p} < \varepsilon/2$$

Therefore $\|f - \varphi\|_{L^p} < \varepsilon$. $\square$