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Lusin's Theorem. Let $f : \mathbb{R}^m \to \mathbb{C}$ be Lebesgue measurable and such that there exists a Lebesgue set A such that

(1) $\mu(A) < \infty$

(2) $f(x) = 0$ if $x \notin A$

then for any $\varepsilon > 0$ there exists $g \in C_c(\mathbb{R}^n)$ such that

$$\mu(\{x : f(x) \neq g(x)\}) < \varepsilon.$$

Moreover one may arrange so that

$$\sup_{x \in \mathbb{R}^m} |g(x)| \leq \sup_{x \in \mathbb{R}^m} |f(x)|.$$

Proof:

Case 1 Suppose that $0 \leq f \leq 1$ and A is compact.

For each n, let

$$S_n(x) = \sum_{j=1}^{2^n} \frac{j-1}{2^n} \chi_{f^{-1}([\frac{j-1}{2^n}, \frac{j}{2^n}))}$$

We know that $0 \le S_1(x) \le S_2(x) \le \cdots$

and $\lim_{n \to \infty} S_n(x) = f(x)$. We can write this

as
$$f(x) = \sum_{n=1}^{\infty} \left(S_n(x) - S_{n-1}(x) \right) \quad ; \quad S_0(x) = 0$$

Let $t_n(x) = S_n(x) - S_{n-1}(x)$.

<u>Lemma:</u> $t_n(x) = 2^{-n} \chi_{T_n}$ where

$$T_n = \bigcup_{\substack{j=1 \\ j \text{ odd}}}^{2^n - 1} f^{-1}\left(\left[\frac{j}{2^n} ; \frac{j+1}{2^n} \right) \right)$$

<u>Proof:</u>

<u>$n = 1$</u>: $S_0(x) = 0$; $S_1(x) = \frac{1}{2} \chi_{f^{-1}([\frac{1}{2}, 1))}$

<u>$n = 2$</u>: $S_2(x) = \frac{1}{4} \chi_{f^{-1}([1/4, 2/4))} + \frac{2}{4} \chi_{f^{-1}([2/4, 3/4))}$

$$+ \frac{3}{4} \chi_{f^{-1}([3/4, 1))}$$

$$S_2(x) - S_1(x) = \frac{1}{4} \chi_{f^{-1}([1/4, 2/4))} + \frac{1}{4} \chi_{f^{-1}([3/4, 1))}$$

The proof for arbitrary n is the same.

Thus we can write $f(x) = \sum_{n=1}^{\infty} t_n(x)$ and the convergence is uniform.

Since $A \subset \mathbb{R}^m$ is compact, there exists $V \subset \mathbb{R}^m$ open with $\overline{V}$ compact such that $A \subset V$.

Since T_n is a Lebesgue set $n = 1, 2, \ldots$ and $T_n \subset A$, there exist a compact set $K_n \subset T_n$ and an open subset V_n with $T_n \subset V_n \subset V$ such that

$$\mu(V_n \setminus K_n) < \varepsilon/2^n.$$

<u>Claim:</u> Let $K \subset \mathbb{R}^m$ be compact and $U \subsetneq \mathbb{R}^m$ be open with $K \subset U$. There exists a continuous function φ such that $0 \leq \varphi(x) \leq 1$

$\varphi(x) = 1$ if $x \in K$ $\quad \varphi(x) = 0$ if $x \notin U$.

Let $h_n(x)$ be continuous and such that $0 \leq h_n(x) \leq 1$; $h_n(x) = 1$ if $x \in K_n$, $h_n(x) = 0$ if $x \notin V_n$.

Let $g(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x)$.

Since the series converges uniformly g is continuous.

$g(x) = 0$ if $x \notin \bar{V}$ which is compact.

Since $2^{-n} h_n(x) = t_n(x)$ except if $x \in V_n \setminus K_n$,

$$g(x) = f(x) \quad \text{unless} \quad x \in \bigcup_{n=1}^{\infty} (V_n \setminus K_n)$$

But $\mu \left(\bigcup_{n=1}^{\infty} (V_n \setminus K_n) \right) < \varepsilon$.

This proves the result if A is compact and

$0 \leq f \leq 1$.

<u>Case 2</u> $\quad 0 \leq f \leq 1$, but A is not necessarily compact.

But since $\mu(A) < \infty$, there exists K compact with $K \subset A$ and $\mu(A \setminus K) < \varepsilon$. We then apply the previous result to $f(x)\chi_K$ and we find g such that

$$\mu\{x : g(x) \neq f(x)\chi_K(x)\} < \varepsilon.$$

and $g(x) = 0$ if $x \notin \mathcal{O}$ where $\mathcal{O}$ is open and $A \subset \mathcal{O}$ with $\mu(\mathcal{O} \setminus A) < \varepsilon$. Therefore

$$\{x : f(x) \neq g(x)\} \subset \{x : g(x) \neq f(x)\chi_K(x)\}$$
$$\cup \, (A \setminus K) \cup (\mathcal{O} \setminus A)$$ and hence

$$\mu(\{x : f(x) \neq g(x)\}) \leq 3\varepsilon.$$

<u>Case 3</u> A not compact; f not necessarily bounded.

Let $B_k = \{x : |f(x)| > k\}$

$B_1 \supset B_2 \supset \cdots$

Since $f(x) = 0$ if $x \notin A$ and $\mu(A) < \infty$, $\mu(B_1) < \infty$. Hence

$$\lim_{k \to \infty} \mu(B_k) = \mu\left(\bigcap_{k=1}^{\infty} B_k\right) = 0$$

Since $f(x) = f(x) \cdot (1 - \chi_{B_k}(x))$ except when $x \in B_k$ and we can take k large such that $\mu(B_k) < \varepsilon$. Case 3 follows from case 2.

Finally If $\sup_{x \in \mathbb{R}^n} |f(x)| = \infty$ we have nothing to prove.

Suppose $R = \sup_{x \in \mathbb{R}^n} |f(x)|$.

Let $\varphi(z) = z$ if $|z| \leq R$

$\varphi(z) = R \dfrac{z}{|z|}$ if $|z| > R$.

$\varphi : \mathbb{C} \to \mathbb{C}$ is continuous and $|\varphi(z)| \leq R$.

If g satisfies

$$\mu(\{x : f(x) \neq g(x)\}) < \varepsilon$$

$$\mu(\{\varphi \circ g(x) \neq f(x)\}) < \varepsilon \quad \text{and}$$

$$\sup |\varphi \circ g| \leq R = \sup |f(x)|.$$

<u>Corollary</u>: Suppose $f : \mathbb{R}^m \to \mathbb{C}$ is measurable, $f(x) = 0$ if $x \notin A$ and $\mu(A) < \infty$. Moreover assume that $|f(x)| \leq 1$. Then there exists a sequence $g_n \in C_c^0(\mathbb{R}^m)$ with $|g_n| \leq 1$ and $f(x) = \lim_{n \to \infty} g_n(x)$ a.e

Proof. For every n $\exists$ $g_n \in C_c(\mathbb{R}^m)$ such that $|g_n| \le 1$ and if

$$E_n = \{ x : f(x) \neq g_n(x) \} \; ; \quad \mu(E_n) < 2^{-n}.$$

$$\sum_{n=1}^{\infty} \mu(E_n) < 1 .$$

Let $g(x) = \sum_{n=1}^{\infty} \chi_{E_n}(x) \qquad x \in \mathbb{R}^m$

$$\int g \, d\mu = \sum \mu(E_n) < \infty$$

So $\mu\{ x : g(x) = \infty \} = 0$, therefore

$$\mu(\{ x : x \in \text{infinitely many } E_n \}) = 0$$

Conclusion: Except for a set of finite measure, every point in $\mathbb{R}^m$ is in at most finitely many E_n

and therefore fixed x

$f(x) = g_n(x)$ if n is large enough.