The space $L^\infty(X,\mu)$

$(X, \mathcal{M}, \mu)$ a measure space

Def: Let $g: X \longrightarrow [0,\infty]$ be measurable.

$$S = \{ \alpha \in [0,\infty] : \mu(g^{-1}(\alpha,\infty]) = 0 \}$$

If $S \neq \phi$, let $\beta = \inf S$. If $S = \phi$, $\beta = \infty$

$\beta =$ essential supremum of g.

If $f: X \longrightarrow \overline{\mathbb{C}}$ is measurable, we define

$$\|f\|_\infty = \beta(|f|)$$

Notice that if $\|f\|_\infty = 0$, then $f = 0$ a.e.

Define

$$L^\infty(X,\mu) = \{ [f] : \|f\|_\infty < \infty \}$$

$[f] = \{ g : g \sim f \}$ $g \sim f$ if and only if $g = f$ a.e.

Proposition: If $f, g \in L^\infty(X,\mu)$; $\lambda \in \mathbb{C}$

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \quad \text{and} \quad \|\lambda f\|_\infty = |\lambda| \cdot \|f\|_\infty$$

<u>Proof</u>: We need to show that

$$(*) \quad \mu(\{x : |f(x) + g(x)| > \|f\|_\infty + \|g\|_\infty\}) = 0$$

But

$$\{x : |f(x) + g(x)| > \|f\|_\infty + \|g\|_\infty\} \subset$$

$$\{x : |f(x)| + |g(x)| > \|f\|_\infty + \|g\|_\infty\}$$

$$\subset \{x : |f(x)| > \|f\|_\infty\} \cup \{x : |g(x)| > \|g\|_\infty\}$$

Hence $(*)$ follows. and the proposition. The second statement is obvious.

<u>Theorem</u>: $L^\infty(X, \mu)$ is a complete metric space.

<u>Proof</u>. Let $f_n \in L^\infty(X, \mu)$ be a Cauchy sequence. ~~[illegible]~~ That is, for every $\varepsilon > 0$ $\exists$ $N = N(\varepsilon)$ such that

$$\|f_m - f_n\| < \varepsilon \quad \text{if} \quad m, n \geq N(\varepsilon)$$

Let $A_n = \{x : |f_n(x)| \geq \|f_n\|_\infty\}$, $\mu(A_n) = 0$

$B_{m,n} = \{x : |f_m(x) - f_n(x)| > \|f_m - f_n\|_\infty\}$

$\mu(B_{m,n}) = 0$

Let $E = \bigcup_{m,n=1}^{\infty} A_n \cup B_{m,n}$, $\mu(E) = 0$.

If $x \notin E$, the sequence $\{f_n(x)\}$ is Cauchy and therefore converges. Moreover the convergence is uniformly. Let $\lim_{n\to\infty} f_n(x) = f(x)$, $x \notin E$

$$\tilde{f}(x) = \begin{cases} f(x), & x \notin E \\ 0, & x \in E \end{cases}$$

But if $m, n \geq N(\epsilon)$; $|f_n(x) - f_m(x)| < \epsilon$ $\forall x \notin E$, and hence $|f(x) - f_n(x)| < \epsilon$ $\forall x \notin E$.

$$\Rightarrow \quad \sup_{x \in E} |f(x)| \leq \epsilon + \|f_n\|_\infty .$$

therefore $f \in L^\infty(X)$ and.

$$\|f - f_n\|_\infty \to 0$$