MA 544 10/2/2009

Jensen's Inequality: $(X, \mathfrak{M}, \mu)$ a measure space. $\Omega \in \mathfrak{M}$. with $\mu(\Omega) = 1$. If $f : X \to (a,b)$ $-\infty \leq a < b \leq \infty$; $f \in L^1(X, \mu)$, and φ is convex on (a,b), then

$$\varphi\left(\int_\Omega f \, d\mu\right) \leq \int_\Omega \varphi \circ f \, d\mu$$

Proof: Let $t = \int_\Omega f \, d\mu$, $-\infty < t < \infty$ since $f \in L^1(X, d\mu)$. Moreover $t \in (a,b)$ since $a < f < b$ and $\mu(\Omega) = 1$.

Let

$$\beta = \sup_{a<s<t} \frac{\varphi(t) - \varphi(s)}{t-s} \leq \frac{\varphi(u) - \varphi(t)}{u-t}$$

provided $s < t < u < b$.

Hence for any $u \in (t,b)$

(1) $\varphi(u) \geq \varphi(t) + \beta(u-t)$.

But on the other hand $\dfrac{\varphi(t) - \varphi(s)}{t-s} \leq \beta$

and so

$$\varphi(s) \geq \varphi(t) + \beta(s-t) \qquad (2)$$

provided $s \in (a, t)$. Therefore putting (1) and (2) together.

$$\varphi(s) \geq \varphi(t) + \beta(s-t) \quad \text{for all } s \in (a,b).$$

In particular, if $x \in \Omega$, $f(x) \in (a,b)$, and thus

$$\varphi(f(x)) \geq \varphi(t) + \beta(f(x) - t)$$

Integrating this inequality we obtain over Ω

$$\int_\Omega \varphi(f(x))\, d\mu \geq \varphi\left(\int_\Omega f\, d\mu\right) + \beta\left(\int_\Omega f\, d\mu - t\right)$$

$\boxed{//}$.

— " —

L^p **spaces, $p > 1$** We want to show that

$$\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p}$$

is a norm on $L^p(X,\mu)$.

It is easy to see that

$$\|\lambda f\|_p = |\lambda| . \|f\|_p \quad \forall f \in L^p(X,\mu).$$

The hard part is to prove the triangle inequality:

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

For this we need a Lemma:

<u>Lemma</u>: (Hölder's Inequality) Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g be measurable functions. Then.

$$\int_X |fg| \, d\mu \leq \|f\|_p \cdot \|g\|_q$$

<u>Proof</u>: If $\|f\|_p = 0$, then $f = 0$ a.e

$\|g\|_q = 0$ $\quad g = 0$ a.e

So it is trivial. If ~~$\|f\|_p \neq \infty$ / and $\|g$~~

So we may assume $\|f\|_p > 0$, $\|g\|_q > 0$.

If either $\|f\|_p = \infty$ or $\|g\|_q = \infty$.

The inequality is also trivial. So we will assume $\|f\|_p < \infty$ and $\|g\|_q < \infty$.

Let

$$F = \frac{|f|}{\|f\|_p} \quad \text{and} \quad G = \frac{|g|}{\|g\|_p}.$$

Next we use the convexity of the exponential function $\varphi(x) = e^x$. If $\alpha > 0$, $\beta > 0$ we write

$$\alpha = e^{s/p} \qquad \beta = e^{t/q} \qquad \text{for } s, t \in \mathbb{R}.$$

$$\alpha\,\beta = e^{s/p} \cdot e^{t/q} = e^{(s/p + t/q)}$$

Since $1/p + \frac{1}{q} = 1$, the convexity of $\varphi = e^x$ implies that

$$e^{s/p + t/q} \leq \frac{1}{p} e^s + \frac{1}{q} e^t = \frac{1}{p} \alpha^p + \frac{1}{q} \beta^q.$$

So if $\alpha, \beta \geq 0$

$$\alpha\,\beta \leq \frac{1}{p} \alpha^p + \frac{1}{q} \beta^q.$$

If we apply this to F. G we obtain

$$F(x) \cdot G(x) \leq \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q \quad \text{a.e}$$

($F(x)$ and $G(x)$ are finite a.e)

If we integrate this in X we obtain

$$\int_X F(x)\, G(x)\, dx \leq \frac{1}{p} \int F^p d\mu + \frac{1}{q} \int G^p d\mu$$

$$= \frac{1}{p} + \frac{1}{q} = 1.$$

So we conclude that

$$\int_X |f g|\, d\mu \leq \|f\|_p \cdot \|g\|_q$$

<u>Theorem:</u> (Minkowski's Inequality)

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p .$$

<u>Proof:</u> If either $\|f\|_p = \infty$ or $\|g\|_p = \infty$, the inequality is trivial. So we assume that both are finite. In particular

$$|f(x)| < \infty \quad \text{and} \quad |g(x)| < \infty \quad \text{a.e.}$$

Next we work

$$(|f| + |g|)^p = (|f| + |g|)(|f| + |g|)^{p-1}$$

$$(|f|+|g|)^p = |f|\,(|f|+|g|)^{p-1} + |g|\,(|f|+|g|)^{p-1}$$

and apply Hölder's inequality. $\frac{1}{p}+\frac{1}{q}=1$; $\frac{1}{q} = 1-\frac{1}{p} = \frac{p-1}{p}$

so $q = \frac{p}{p-1}$ $\qquad q(p-1) = p$

$$\int (|f|+|g|)^p \, d\mu \leq \|f\|_p \cdot \left(\int (|f|+|g|)^p \, d\mu \right)^{\frac{p-1}{p}}$$

$$+ \|g\|_p \cdot \left(\int (|f|+|g|)^p \, d\mu \right)^{\frac{p-1}{p}}$$

Next we want to divide the inequality

by $\left(\int (|f|+|g|)^p \, d\mu \right)^{\frac{p-1}{p}}$. But to do

that we need to verify that it is finite.

This is a consequence of the convexity

of the function $\varphi(t) = t^p$, $p>1$

$$\left(\frac{|f|+|g|}{2} \right)^p \leq \frac{1}{2}|f|^p + \frac{1}{2}|g|^p$$

and ~~keen~~ since $\|f\|_p < \infty$ and $\|g\|\infty$

and $1 - \frac{p-1}{p} = \frac{1}{p}$, we obtain the

result. $\square$.

<u>Theorem:</u> Let (X, μ) be a measure space and $p \in (1, \infty)$. and define.

$$L^p(X, d\mu) = \left\{ [f] : \|f\|_p = \left(\int |f|^p d\mu \right)^{1/p} < \infty \right\}.$$

Then $L^p(X, d\mu)$ is a complete vector space.

<u>Proof.</u>

$$\left.\begin{aligned} \|f+g\|_p &\le \|f\|_p + \|g\|_p \\ \|\alpha f\|_p &= |\alpha| \cdot \|f\|_p \end{aligned}\right\} \begin{array}{l} L^p(X, d\mu) \\ \text{is a vector} \\ \text{space over } \mathbb{C} \end{array}$$

Let $\{f_n, n \in \mathbb{N}\}$ be a Cauchy sequence in L^p. Just as in the case of $L^1(\mathbb{R}^n)$ we construct a subsequence f_{n_k}, with $n_1 < n_2 < n_3 < \ldots$ such that

$$\| f_{n_{k+1}} - f_{n_k} \| \le \frac{1}{2^k} \qquad k = 1, 2, \ldots$$

Let

$$g_K = \sum_{k=1}^{K} | f_{n_{k+1}} - f_{n_k} | ; \quad g = \sum_{k=1}^{\infty} | f_{n_{k+1}} - f_{n_k} |$$

g_K are measurable functions and

$$g_K \leq g_{K+1}$$

From Minkowski's inequality

$$\| g_K \|_{L^p} \leq \sum_{k=1}^{K} \| f_{n_{k+1}} - f_{n_k} \|_{L^p} < 1.$$

Hence.

$$\int_X | g_K |^p \, d\mu < 1$$

take the limit as $K \to \infty$ and use the monotone convergence theorem.

$$\int_X |g|^p \, d\mu < 1.$$

In particular $g(x) < \infty$ a.e

hence the series

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$$

converges absolutely a. e.

But notice that

$$\sum_{k=1}^{K-1} \left(f_{n_{k+1}} - f_{n_k} \right) + f_{n_1} = f_{n_K}(x)$$

$$\lim_{n_k \to \infty} f_{n_k}(x) = f(x) \quad \text{a. e.}$$

This is the candidate for the L^p limit of the sequence.

But by the construction

$$f(x) - f_{n_K}(x) = \sum_{k=K}^{\infty} f_{n_{k+1}} - f_{n_k}.$$

Again by Minkowski's inequality and monotone convergence theorem

$$\| f - f_{n_K} \|_{L^p} \to 0 \qquad \boxed{///}$$

Corollary: Let $p \in [1, \infty)$. and let $\{f_n, n \in \mathbb{N}\}$ be a Cauchy sequence in $L^p(X, d\mu)$. ~~Then~~ with limit f. Then there exists a subsequence f_{n_k} such that

$$\lim_{n_k \to \infty} f_{n_k}(x) = f(x) \quad \text{a.e.}$$