Let $(X, \mathcal{S}, \mu)$; (Y, τ, ν) be σ-finite measure spaces. We proved that $\forall\, Q \in \mathcal{S} \times \tau$

$$I(Q) = \int_X \left(\int_Y \chi_Q(x,y)\, d\nu \right) d\mu = \int_Y \left(\int_X \chi_Q(x,y)\, d\mu \right) d\nu$$

Therefore we define $(\mu \times \nu)(Q) = I(Q)$.

Proposition: $\mu \times \nu$ is a measure on $\mathcal{S} \times \tau$.

Proof: Let $Q_j \in \mathcal{S} \times \tau$, $j = 1, 2, \ldots$ be such that $Q_j \cap Q_k = \emptyset$. In this case if $Q = \bigcup_{j=1}^{\infty} Q_j$

$$\chi_Q = \sum_{j=1}^{\infty} \chi_{Q_j}(x,y)$$

Fixed N, let $\tilde{Q}_N = \bigcup_{j=1}^{N} Q_j$. Then

$$\mu \times \nu(\tilde{Q}_N) = \int_X \left(\int_Y \chi_{\tilde{Q}_N}(x,y)\, d\nu \right) d\mu =$$

$$= \sum_{j=1}^{N} \int_X \left(\int_Y \chi_{Q_j}(x,y)\, d\nu \right) d\mu = \sum_{j=1}^{N} \mu \times \nu(Q_j)$$

Since $\int_Y \chi_{\tilde{Q}_N}(x,y)\, d\nu \le \int_Y \chi_{\tilde{Q}_{N+1}}(x,y)\, d\nu$

The monotone convergence theorem implies that

$$\lim_{N\to\infty} \mu\times\nu(\tilde{Q}_N) = \int_X \lim_{N\to\infty} \left(\int_Y \chi_{\tilde{Q}_N}(x,y)\, d\nu \right) d\mu$$

Again, by the monotone convergence theorem, since $\chi_{\tilde{Q}_N} \le \chi_{\tilde{Q}_{N+1}}$ we obtain.

$$\lim_{N\to\infty} \int_Y \chi_{\tilde{Q}_N}(x,y)\, d\nu = \int_Y \chi_Q(x,y)\, d\nu.$$

Therefore

$$\lim_{N\to\infty} \mu\times\nu(\tilde{Q}_N) = \int_X \left(\int_Y \chi_Q(x,y)\, d\nu \right) d\mu. \quad \square$$

<u>Theorem:</u> Let $f : X\times Y \longrightarrow [0,\infty]$ be $\mathcal{S}\times\mathcal{T}$ measurable. Then

$\varphi(x) = \int_Y f_x(y)\, d\nu$ is $\mathcal{S}$-measurable

$\psi(y) = \int_X f^y(x)\, d\mu$ is $\mathcal{T}$-measurable

and
$$\int_X \varphi \, d\mu = \int_Y \psi \, d\nu = \int_{X\times Y} f \, d(\mu\times\nu)$$

Proof. This result is true for $f = \chi_Q$, $Q \in \mathcal{S}\times\mathcal{T}$.

In general, we take a sequence $s_n(x,y)$ of simple function satisfying

$$0 \le s_1 \le s_2 \le \dots \quad \text{and such that } \lim_{N\to\infty} s_N = f$$

The monotone convergence theorem implies.

$$\lim_{n\to\infty} \int_Y s_n(x,y)\, d\nu = \int_Y f(x,y)\, d\nu = \varphi(x)$$

So φ is $\mathcal{S}$-measurable. Similarly one proves that ψ is $\mathcal{T}$-measurable. Again the monotone convergence theorem (applied twice) gives

$$\lim_{n\to\infty} \int_X \left(\int_Y s_n(x,y)\, d\nu \right) d\mu = \lim_{n\to\infty} \int_Y \left(\int_X s_n(x,y)\, d\mu \right) d\nu$$

$$= \int_X \varphi(x)\, d\mu = \int_Y \psi(y)\, d\mu = \int_{X\times Y} f \, d(\mu\times\nu).$$

Corollary: Let $f: X \times Y \longrightarrow \mathbb{C}$ be measurable.

A) $f \in L^1(\mu \times \nu) \iff \varphi(x) = \int_Y |f_x| \, d\nu \in L^1(\mu)$ or $\psi(y) = \int_X |f^y| \, d\mu \in L^1(\nu)$

B) If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ a.e x,

$f^y \in L^1(\mu)$ a.e y and

$$(*) \quad \int_{X \times Y} f \, d(\mu \times \nu) = \int_{X} \left(\int_Y f(x,y) \, d\nu \right) d\mu = \int_Y \left(\int_X f(x,y) \, d\mu \right) d\nu.$$

Proof: (A) If one applies the theorem to $|f|$ one obtains

$$\int_{X \times Y} |f| \, d(\mu \times \nu) = \int_X \varphi(x) \, d\mu = \int_Y \psi(y) \, d\nu.$$

(B) Since $f \in L^1(\mu \times \nu)$, the theorem implies that

$$\int |f| \, d(\mu \times \nu) = \int_Y \left(\int_X |f^y| \, d\mu \right) d\nu.$$ In particular

this shows that for almost every y $\int_X |f^y| \, d\mu < \infty$

~~thus~~ i.e $f^y \in L^1(\mu)$ a.e. y.

Similarly $f_x \in L^1(\nu)$ a.e x.

To prove (*) we first assume that f is real valued. Then $f = f_+ - f_-$ with

$$f_+ = \max(0, f), \quad f_- = \max(0, -f).$$

Of course $f_\pm \le |f|$. The theorem applied to f_+ and f_- gives

$$\int_X \left(\int_Y f_+(x,y)\, d\nu \right) d\mu = \int_{X\times Y} f_+ \, d(\mu\times\nu) < \int |f| \, d(\mu\times\nu) < \infty$$

Similarly for f_-. Therefore.

$$\int_X \left(\int_Y f(x,y)\, d\nu \right) d\mu = \int_Y \left(\int_X f(x,y)\, d\mu \right) d\nu =$$

$$= \int_{X\times Y} f \, d(\mu\times\nu). \quad \square.$$