MA 544 10/05/09

Notions of Convergence

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_n : X \longrightarrow [-\infty, \infty]$, $n \in \mathbb{N}$, a sequence of measurable functions.

1. Convergence almost everywhere: $f_n \to f$ a.e in $A \in \mathcal{M}$ if there exists $B \subset A$, $B \in \mathcal{M}$, such that $f_n(x) \longrightarrow f(x)$ $\forall\, x \in B$ and $\mu(B \setminus A) = 0$.

• Notice that if $f_n \longrightarrow f$ a.e in A and $f_n \longrightarrow g$ a.e in A, then $f = g$ a.e in A.

2. Convergence in the mean: We say that $f_n \longrightarrow f$ in the mean if (f, f_n are finite) and

$$\lim_{n \to \infty} \int_X |f - f_n| \, d\mu = 0$$

(one can assume f_n, f are finite a.e only)

If $f_n \to f$ and $f_n \to g$ in the mean, then $f = g$ a.e.

__Def:__ We say that $\{f_n\}$ is Cauchy in the mean if f_n is finite and $\forall\ \varepsilon > 0\ \exists\ N$ s.t

$$\int |f_m - f_n|\, d\mu < \varepsilon \quad \text{if} \quad m, n \geq N.$$

__Theorem:__ If f_n is Cauchy in the mean, then there exists $f : X \to (-\infty, \infty)$ such that $f_n \to f$ in the mean and there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e.

3. __Convergence in Measure__ $f_n \to f$ in measure if f is finite and for every $\varepsilon > 0$

$$\lim_{n \to \infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$$

Similarly we say that $\{f_n\}$ is Cauchy in measure if $\forall\ \varepsilon > 0$

$$\lim_{m,n \to \infty} \mu(\{x \in X : |f_m(x) - f_n(x)| \geq \varepsilon\}) = 0.$$

Proposition: If $f_n \to f$ and $f_n \to g$ in measure. then $f = g$ a.e.

Proof: Exercise.

Theorem: If $\{f_n\}$ is Cauchy in measure on X, then exists $f : X \longrightarrow (-\infty, \infty)$ such that $f_n \to f$ in measure. Moreover, there is a subsequence $\{f_n\}$ which converges to f a.e.

Proof. We have for all k

$$\lim_{m,n \to \infty} \mu\left(\{x \in X \mid |f_m(x) - f_n(x)| \geq 1/2^k\}\right) = 0$$

therefore there exists a sequence $n_1 < n_2 < \cdots$ such that

$$\mu\left(\{x \in X \mid |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 1/2^k\}\right) < 1/2^k.$$

$$E_k = \{x \in X : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 1/2^k\}$$

$$\mu(E_k) < 1/2^k$$

$$F_m = \bigcup_{k=m}^{\infty} E_k \ ; \qquad F = \bigcap_{m=1}^{\infty} F_m$$

$$\lim_{k\to\infty} \mu(E_k) = 0$$ Does this imply that

$$\lim_{k\to\infty} \chi_{E_k} = 0 \quad \text{a.e ?}$$

$$\limsup_{k\to\infty} \chi_{E_k} = \lim_{k\to\infty} \sup_{m\geq k} \chi_{E_m}$$

$$\sup_{m\geq k} \chi_{E_k} = \chi_{F_m} \qquad F_m = \bigcup_{k=m}^{\infty} E_k$$

Notice that $F_1 \supset F_2 \supset \cdots$

$$\lim_{m\to\infty} \chi_{F_m} = \chi_F \qquad F = \bigcap_{m=1}^{\infty} F_m$$

Since $\mu(F_m) \leq \sum_{k=m}^{\infty} \mu(E_k) \leq \frac{1}{2^{m-1}}.$

$\mu(F) = 0.$

$$\mu(F_m) \le \sum_{k=m}^{\infty} \mu(E_k) < \frac{1}{2^{m-1}}$$ and therefore

$$\mu(F) = 0.$$

If $x \notin F$, $\exists\, m$ such that $x \notin F_m$ and therefore

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| < 1/2^k \quad \forall\, k \geq m$$

and

$$\sum_{k=m}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < \frac{1}{2^{m-1}}$$

So the series

$$\sum_{k=m}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x) \qquad m = 1, 2 \ldots$$

converges and we define

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x). \qquad x \notin F$$

The argument above shows that $\lim_{n_k \to \infty} f_{n_k}(x) = f(x)$, $x \notin F$.

Now we want to show that $f_n \to f$ in measure.

If $x \notin F_m$, then

$$|f_{n_m}(x) - f(x)| \leq \sum_{k=m+1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < \frac{1}{2^{m-1}}$$

So ~~the~~

$$\{x : |f(x) - f_{n_m}(x)| \geq \frac{1}{2^{m-1}}\} \subset F_m$$

and therefore

$$\mu(\{x : |f(x) - f_{n_m}(x)| \geq \frac{1}{2^{m-1}}\}) \leq \mu(F_m) < \frac{1}{2^{m-1}}$$

Fixed $\varepsilon > 0$, pick m so that $\frac{1}{2^{m-1}} < \varepsilon$

then

$$\{x : |f(x) - f_{n_m}(x)| \geq \varepsilon\} \subset \{x : |f(x) - f_{n_m}(x)| \geq \frac{1}{2^{m-1}}\}$$

$$< \frac{1}{2^{m-1}}$$

Therefore $f_{n_m} \to f$ in measure.

To conclude that $f_n \to f$ in measure.

$$|f_n(x) - f(x)| \leq |f_n(x) - f_{n_k}(x)| + |f_{n_k}(x) - f(x)|$$

So v/4

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\} \subset \{x : |f_n(x) - f(x)| \geq \varepsilon/2\} \cup$$

$$\{x : |f_{n_k}(x) - f(x)| \geq \varepsilon/2\}$$

and hence

$$\lim_{n \to \infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0 \quad \forall \varepsilon > 0.$$