MA 544 9/18/09

The Lebesgue Measure

Step 1: We choose a family $\mathcal{F} := \{\phi, X_j, j \in \mathbb{N}\}$

$X_j \subset \mathbb{R}^n$ such that

$V(X_j) =$ Volume of X_j is well defined.

This gives a map

$$V : \mathcal{F} \longrightarrow [0, \infty]$$

Step 2: We construct an outer measure on $\mathbb{R}^n$ out of $\mathcal{F}$ and V such that $\forall E \subset \mathbb{R}^n$

$$\mu^*(E) = \inf \left\{ \sum_j V(E_j) : E_j \in \mathcal{F},\ E \subset \bigcup_{j=1}^{\infty} E_j \right\}$$

Step 3: We construct a σ-algebra and a measure by setting

$$\Sigma_* := \{ A \subset \mathbb{R}^n : \mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E) \ \forall E \subset \mathbb{R}^n \}$$

then for every $A \in \Sigma_*$

$$\mu(A) = \mu^*(A)$$

μ is a measure.

Of course there are infinitely many choices of the family $\{X_i\}$ and therefore infinitely many possible measures.

<u>Step 4</u> However we want to define a measure which would extend the notion of volume. therefore we want the final measure μ to satisfy

(1) If $I = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$

we want $\mu(I) = \prod_{j=1}^{n} (b_j - a_j)$

(2) All Borel sets are contained in Σ_*

(3) If $K \subset \mathbb{R}^n$ is compact $\mu(K) < \infty$

(4) μ is translation invariant.

We begin by choosing an appropriate family $\mathcal{F} = \{X_j\}$.

A closed interval $I \subset \mathbb{R}^n$ is a set of the form

$$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

$$-\infty < a_j < b_j < \infty$$

an open interval

$$I = (a_1, b_1) \times \cdots \times (a_n, b_n) \qquad -\infty < a_j < b_j < \infty$$

$$V(I) = \prod_{j=1}^{n} (b_j - a_j)$$

Covering Lemma: Let $\mathcal{O} \subset \mathbb{R}^n$ be an open subset, $\mathcal{O} \neq \emptyset$. Then there exist closed intervals I_j, $j \in \mathbb{N}$, such that

$$(1)\ \ \mathring{I}_j \cap \mathring{I}_k = \emptyset \qquad (2)\ \ \mathcal{O} = \bigcup_{j=1}^{\infty} I_j$$

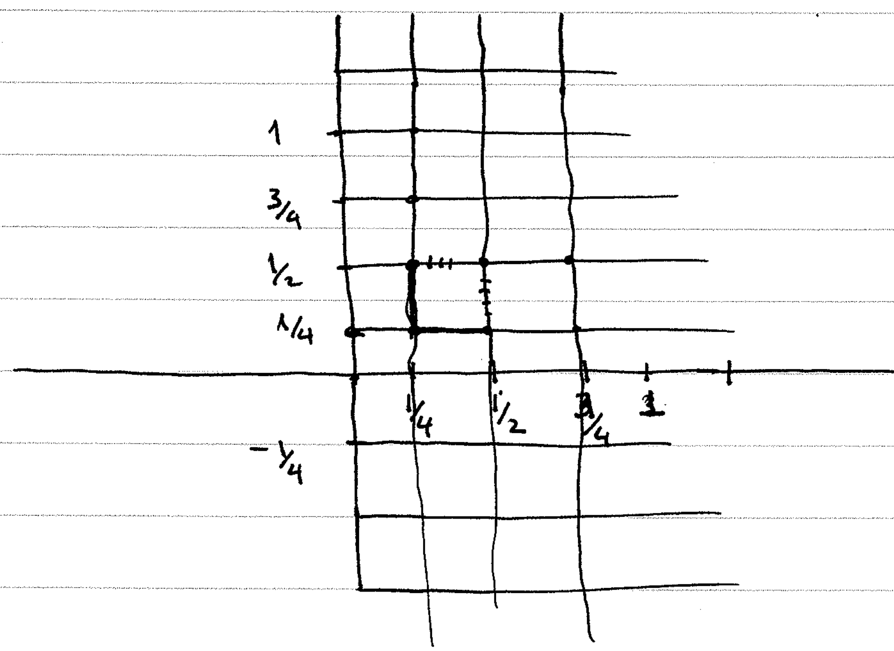
Proof: Fix k and let

$$P_k = \{ (p_1, \dots, p_n) : p_j = m\, 2^{-k},\ m \in \mathbb{Z} \}$$

For $p \in P_k$, define

$$I(p, 2^{-k}) = \{ x \in \mathbb{R}^n : p_j \le x_j < p_j + 2^{-k},\ 1 \le j \le n \}$$

$k = 2$



$$\Omega_k = \{ I(p, 2^{-k}) : p \in P_k \}$$

These sets satisfy the following:

(1) Fixed k, then for each $x \in \mathbb{R}^n$, there exists a <u>unique</u> interval $I(p, 2^{-k})$ with

$$x \in I(p, 2^{-k})$$

(2) Let $k' > k$. If $I(p_1, 2^{-k}) \in \Omega_k$ and $I(p_2, 2^{-k'}) \in \Omega_{k'}$ then either

$$I(p_2, 2^{-k'}) \subset I(p_1, 2^{-k})$$ or

$$I(p_2, 2^{-k'})^{\circ} \cap I(p_1, 2^{-k}) = \phi$$

Since $\mathcal{O}$ is open, then for every $x \in \mathcal{O}$ there exists k and $p \in P_k$ such that

$$x \in I(p, 2^{-k}) \subset \mathcal{O}$$

$$\mathcal{F} = \{ I(p, 2^{-k}) \in \Omega_k ; \ \bar{I}(p, 2^{-k}) \subset \mathcal{O} \}$$

$$\mathcal{O} = \bigcup_{\mathcal{F}} I(p, 2^{-k})$$

Now we filter the collection by doing the following:

(1) Keep all the intervals in Ω_1 and discard all the others in Ω_k, $k > 1$ which are contained in one of these.

(2) Keep all the intervals in Ω_2 which are not contained in the ones in Ω_1 from step 1. Discard the ones in Ω_k, $k > 2$, contained in these.

(3) Repeat the process $n = 3, 4 \ldots$

The remaining intervals satisfy

$$\mathcal{O} = \bigcup_{j=1}^{\infty} I_j \qquad \bar{I}_j \subset \mathcal{O}$$

and $I_j \cap I_k = \phi$

To finish the proof just pick the closure of these intervals

We will work with the family

$$\mathcal{F} = \{ \Omega_k, \ k = 1, 2, \ldots \} \cup \{ \phi \}.$$