MA 544 9/28/09

Theorem: Let $f : [a,b] \to \mathbb{R}$ be bounded and suppose $f \in \mathcal{R}([a,b])$. Then f is Lebesgue measurable and

$$\int_a^b f\,dx = \int_{[a,b]} f\,d\mu.$$

Proof: For $n \in \mathbb{N}$; divide the interval $[a,b]$ into n-equal parts: $\Delta_n = \frac{b-a}{n}$; $x_{j,n} = a + j\Delta_n$ $j = 0, 1, \dots, n$.

Let $\mathcal{P}_n = \{x_0 = a < x_1 < x_2 < \dots < x_n = b\}$. We see that

$$\mathcal{P}_{n+1} \subset \mathcal{P}_n, \quad n = 1, 2 \dots$$

Let $I_{j,n} = [x_{j,n}; x_{j+1,n}]$ $\qquad m_{j,n} = \inf_{I_{j,n}} f$

$\qquad M_{j,n} = \sup_{I_{j,n}} f$.

$$L_n(x) = \sum_{j=0}^{n} m_{j,n} \chi_{I_{j,n}}.$$

$$U_n(x) = \sum_{j=0}^{n} M_{j,n} \chi_{I_{j,n}}.$$

$$L_n(x) \le f(x) \le U_n(x) \qquad n = 1, 2, \dots$$

The sequences $L_n(x)$ and $U_n(x)$ are monotone (L_n is increasing and U_n is decreasing). Let

$$L(x) = \lim_{n\to\infty} L_n(x) \quad \text{and} \quad U(x) = \lim_{n\to\infty} U_n(x).$$

$$L(x) \leq g(x) \leq U(x)$$

But $\int_a^b L_n(x)\,dx = L(P_n)$ and $\int_a^b U_n(x)\,dx = U(P_n)$

$$\int_a^b L_n(x)\,dx = \int_{[a,b]} L_n\,d\mu \; ; \qquad \int_a^b U_n(x)\,dx = \int_{[a,b]} U_n\,d\mu .$$

By the ~~m~~ monotone convergence theorem.

$$\int_{[a,b]} L\,d\mu = \lim_{n\to\infty} \int_{[a,b]} L_n\,d\mu$$

$$\int_{[a,b]} U\,d\mu = \lim_{n\to\infty} \int_{[a,b]} U_n\,d\mu$$

But since $f \in R([a,b])$.

$$\int_a^b f\,dx = \lim_{n\to\infty} \int L_n\,dx = \lim_{n\to\infty} \int U_n\,dx$$

Therefore

$$\int_{[a,b]} (U - L)\, d\mu = 0 \implies U = L \quad a.e$$

Since $L(x) \le g(x) \le U(x)$ we find that

$$g = U \quad a.e.$$

Hence g is Lebesgue measurable and

$$\int_{[a,b]} g \, d\mu = \int_{[a,b]} U \, d\mu = \int_a^b g(x)\, dx.$$