MA544 LECTURE NOTES, FALL 2009

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1. Introduction

These are my lecture notes for MA 544, fall 2009. They were actually typed in the fall 2005 and have not been changed, even though it obviously contains several typos. But I guess they are helpful. Here is hw 1 for the course: Give me a list of typos.

There are no required text books. I will use a bit of each of the following references, which have been placed on reserve in the library. .

- A. Friedman, Foundations of Modern Analysis. Dover Edition, 1982
- W. Rudin, Principles of Mathematical Analysis, 3rd Ed. McGraw-Hill, 1986
- W. Rudin, Real and Complex Analysis, 3rd Ed. McGraw-Hill, 1986
- H.L. Royden, Real Analysis, Prentice-Hall, 1988
- A. Torchinsky, Real Variables. Addison-Wesley, 1988

2. Preliminaries: Properties of $\mathbb{R}^n$, Compact Sets, Continuous Functions, etc.

We review some properties of the n-dimensional Euclidean space $\mathbb{R}^n$ and its topology.

Definition 2.1. *We denote* $\mathbb{R}$ *the set of real numbers. The Euclidean space* $\mathbb{R}^n$ *is defined as*

$$\mathbb{R}^n = \{x = (x_1, ..., x_n) : \quad x_j \in \mathbb{R}.\}$$

As we all know, $\mathbb{R}^n$ is a normed vector space. One norm of an element $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is

$$||x|| = \left(x_1^2 + ... + x_n^2\right)^{\frac{1}{2}}$$

The distance between two points $x, y \in \mathbb{R}^n$ is

$$d(x, y) = ||x - y||.$$

We then define open and closed subsets:

Definition 2.2. *A subset* $X \subset \mathbb{R}^n$ *is open if for every* $x \in X$ *there exists an* $\epsilon > 0$ *such that*

$$B(x, \epsilon) = \{y \in \mathbb{R}^n : ||y - x|| < \epsilon\} \subset X.$$

A subset $F \subset \mathbb{R}^n$ *is closed if* $\mathbb{R}^n \setminus F$ *is open.*

Exercise 2.1. If $\{U_\alpha, \ \alpha \in A\}$ is a family of open subsets of $\mathbb{R}^n$, show that $\bigcup_\alpha U_\alpha$ is open. Similarly show that if If $\{F_\alpha, \ \alpha \in A\}$ is a family of closed subsets of $\mathbb{R}^n$, show that $\bigcap_\alpha F_\alpha$ is closed.

The family of all open sets of $\mathbb{R}^n$ is described as a topology on $\mathbb{R}^n$, and $\mathbb{R}^n$ equipped with this family of open sets is called a topological space.

A subset $U \subset \mathbb{R}^n$ can be equipped with what is called the relative topology. We say that a subset $O \subset U$ is open with respect to the relative topology if $O = X \cap U$ where $X \subset \mathbb{R}^n$ is open.

Definition 2.3. *A family* $\{U_\alpha,\ \alpha \in A\}$ *of open subsets of* $\mathbb{R}^n$ *is said to be an open cover of a subset* $K \subset \mathbb{R}^n$ *if*

$$K \subset \bigcup_{\alpha \in A} U_\alpha.$$

An open cover $\{U_\alpha,\ \alpha \in A\}$ *of* K *is said to have a finite subcover if there exists* $U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_N}$, *with* $\alpha_1, ..., \alpha_N \in A$ *such that*

$$K \subset \bigcup_{j=1}^{n} U_{\alpha_j}.$$

Theorem 2.1. *Let* $K \subset \mathbb{R}^n$. *The following properties of* K *are equivalent:*
1) K *is bounded and closed*
2) Every open cover of K *has a finite subcover*
3) Every sequence of points of K *has a subsequence that converges to a point of* K.

Proof. $1 \implies 2$ Since K is closed, $\mathbb{R}^n \setminus K$ is open. Since K is bounded, then there exist $a, b \in \mathbb{R}$, $a < b < \infty$, such that $K \subset Q$, with $Q = [a, b] \times [a, b] \times ... \times [a, b]$. If $\{U_\alpha,\ \alpha \in A\}$ is an open cover of K, then $(\mathbb{R}^n \setminus K) \cup \{U_\alpha,\ \alpha \in A\}$ is an open cover of Q. Thus we need to prove that an arbitrary open cover of Q has a finite subcover. Suppose, by contradiction, that this is not true, i.e. there exists an open cover $\{U_\alpha\}$ of Q which does not have a finite subcover that covers Q. Let us introduce some important notation: Let $U \subset \mathbb{R}^n$, we define *the diameter of* U *as*

$$\operatorname{diam}(U) = \sup\{||x - y|| : x, y \in U\}.$$

Then $d = \operatorname{diam}(Q) = (b - a)\sqrt{n}$.

Let $c = (b - a)/2$. *The intervals* $[a, c]$ *and* $[b, c]$ *determine* 2^n *cubes* Q_j *with* $\operatorname{diam}(Q_j) = d/2$. *At least one of these sets, let's say,* Q_1 *cannot be covered by a finite subfamily of* $\{U_\alpha\}$. *Then we divide* Q_1 *in* 2^n *cubes of diameter* $d/4$, *and repeat the process. We obtain a family of cubes* $\{B_j,\ \ j \in \mathbb{N}\}$ *such that*

$$\operatorname{diam}(B_j) = \frac{d}{2^j},$$
$$B_1 \supset B_2 \supset B_3....$$
$$B_j \text{ cannot be covered by any finite subfamily of } \{U_\alpha\},\ \ j = 1, 2, ...$$

We then need the following:

Lemma 2.1. *Let* $B_j = [a_{1,j}, b_{1,j}] \times [a_{2,j}, b_{2,j}] \times ...[a_{n,j}, b_{n,j}]$, $B_1 \supset B_2 \supset B_3...$ *then*

$$\bigcap_{j=1}^{\infty} B_j \neq \emptyset.$$

Proof. Let us suppose first that $n = 1$. Denote $a_{1,j} = a_j$, $b_{1,j} = b_j$ and $E = \{a_j,\ j \in \mathbb{N}\}$ be the set of lower endpoints of the intervals. Let $x = \sup E$. If $m, n \in \mathbb{N}$, then

$$a_n \leq a_{m+n} \leq b_{m+n} \leq b_m.$$

Therefore $x \leq b_m$ for every m. Since it is obvious that $x \geq a_m$, we have $x \in I_j$, for all $j \in \mathbb{N}$.

In the general case, for each $k \in \{1, ..., n\}$, there exists $x_k \in \bigcap_{j=1}^{\infty} [a_{k,j}, b_{k,j}]$. Then $(x_1, ..., x_n) \in \bigcap_{j=1}^{\infty} B_j$. □

Let $x^* \in \bigcap_{j=1}^{\infty} B_j$. *Since* $\{U_\alpha\}$ *covers* B_j, $j \in \mathbb{N}$, *there exists* α_0 *such that* $x^* \in U_{\alpha_0}$. *Since* U_{α_0} *is open, there exists* $\delta > 0$ *such that* $x^* \in B(x^*, \delta) = \{y \in \mathbb{R}^n : ||y - x|| < \delta\} \subset U_{\alpha_0}$. *On the other hand, since*

$x^* \in B_j$, $j = 1, 2...$ and $\operatorname{diam}(B_j) = d/2^j$, we can pick j large enough such that $B_j \subset U_{\alpha_0}$. Therefore B_j is covered by the set U_{α_0} and this is a contradiction. This ends the proof of the first implication.

$2 \Longrightarrow 1$ *Suppose every open cover of K has a finite subcover. First we show that K is bounded. Let $\delta > 0$ and for each $x \in K$ let $B(x, \delta) = \{y \in \mathbb{R}^n : ||y - x|| < \delta\}$. The family $\{B(x, \delta) : x \in K\}$ is an open cover of K and therefore has a finite subcover. $B(x_k, \delta)$, $1 \leq k \leq N$. Hence $K \subset \bigcup_{k=1}^{N} B(x_k, \delta)$. It is then easy to see that K is bounded.*

Now we want to show that K is closed. Let $p \in \mathbb{R}^n \backslash K$. For each $q \in K$ let $B(q, r(q))$ be the ball centered in q with radius $r(q) < ||p-q||/4$. This gives an open cover of K. Since K is compact, there exists $q_1, ..., q_N$ such that $K \subset \bigcup_{j=1}^{N} B(q_j, r(q_j)) = W$. Now let $B(p, r(q_j))$ be the ball centered at p with radius $r(q_j)$. It follows from the definition of $r(q)$ that $B(p, r(q)) \cap B(q, r(q)) = \emptyset$. Therefore if $\bigcup_{j=1}^{N} B(p, r(q_j)) = V$ then V is open, $p \in V$, and $V \cap W = \emptyset$. So in particular $V \cap K = \emptyset$. This proves the implication.

$2 \Longrightarrow 3$ *Let $\{y_j,\ j \in \mathbb{N}\}$ be sequence contained in K. For $\delta > 0$ the family $\{B(x, \delta),\ x \in K\}$ is an open cover of K. Hence there exists x_k, $1 \leq k \leq N$, and a subsequence $\{y_{j,n}\} \subset \{y_j\}$ such that $||y_{j,n} - x_k|| < \delta$. Repeating this process, we deduce that $\{y_j\}$ has a Cauchy subsequence $\{y_{j,n}\}$. Therefore it converges. Since K is closed, its limit belongs to K. This proves the implication.*

$3 \Longrightarrow 1$ *If K is not bounded, one can construct a sequence of elements of K that does not converge. 3 also clearly implies that K is closed.*

This ends the proof of the Theorem.

□

There is one result that we will need later, and it is important to review it.

Theorem 2.2. *Let $\{K_\alpha, \quad \alpha \in A\}$ be a family of compacts subset of $\mathbb{R}^n$ such that for any finite subcollection $\{K_{\alpha_j}, \quad 1 \leq j \leq N\}$, $\bigcap_{j=1}^{N} K_{\alpha_j} \neq \emptyset$. Then $\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$.*

Proof. Suppose by contradiction that $\bigcap_{\alpha \in A} K_\alpha = \emptyset$. Fix an element K_{α_1} of the family. By assumption $K_{\alpha_1} \bigcap_{\{\alpha \in A, \alpha \neq \alpha_1\}} K_\alpha = \emptyset$. This is equivalent to saying that if $G_\alpha = \mathbb{R}^n \backslash K_\alpha$. Then $K_{\alpha_1} \subset \bigcup_{\{\alpha \in A, \alpha \neq \alpha_1\}} G_\alpha$ Since G_α is open, and K_{α_1} is compact, there exist $\alpha_2, ..., \alpha_N$ such that $K_{\alpha_1} \subset \bigcup_{j=2}^{N} G_{\alpha_j}$. But this implies that $\bigcap_{j=1}^{N} K_{\alpha_j} = \emptyset$, which is a contradiction. □

Now we discuss continuity.

Definition 2.4. *Let $X \subset \mathbb{R}^n$. We say that a function $f : X \longrightarrow \mathbb{R}$ is continuous at a point $x_0 \in X$, if for every $\epsilon > 0$ there exists a $\delta > 0$, δ depending on ϵ and x_0 such that*

$$\text{for all} \quad x \in X \quad \text{with} \quad ||x - x_0|| < \delta \Longrightarrow |f(x) - f(x_0)| < \epsilon. \tag{2.1}$$

We say that f is continuous at a subest $X \subset \mathbb{R}^n$ if f is continuous at every point of X.

Exercise 2.2. Show that a function $f : X \longrightarrow \mathbb{R}$ is continuous everywhere if and only if for every open subset $V \subset \mathbb{R}$, $f^{-1}(V)$ is an (relatively) open subset of X.

Notice that (2.1) in the definition of continuity involves two inequalities: that is f is continuous at x_0 if and only if

$$||x - x_0|| < \delta \Longrightarrow f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon.$$

This motivates the following definition:

Definition 2.5. *Let $X \subset \mathbb{R}^n$. A function $f : X \longrightarrow \mathbb{R}$ is upper semicontinuous at $x_0 \in X$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$x \in X, \quad ||x - x_0|| < \delta \quad \text{implies} \quad f(x) < f(x_0) + \epsilon. \tag{2.2}$$

Similarly, a function $f : X \longrightarrow \mathbb{R}$ *is lower semicontinuous at* $x_0 \in X$ *if for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$x \in X, \quad ||x - x_0|| < \delta \quad \textit{implies} \quad f(x_0) - \epsilon < f(x). \tag{2.3}$$

A function is upper (or lower) semicontinous in X *if it is upper (or lower) semicontinous at every* $x_0 \in X$.

Theorem 2.3. *Let* $X \subset \mathbb{R}$. *A function* $f : X \longrightarrow \mathbb{R}$ *is lower semicontinuous on* X *if and only if* $f^{-1}((k, \infty))$ *is (a relative) open subset of* X *for all* $k \in \mathbb{R}$. *Similarly* f *is upper semicontinuous if and only if* $f^{-1}((-\infty, k))$ *is (a relative) open subset of* X *for all* $k \in \mathbb{R}$.

Proof. Suppose f is lower semicontinuous in X. Let $k \in \mathbb{R}$ and let $x_0 \in f^{-1}((k, \infty))$. Then $f(x_0) > k$. Let $\epsilon = f(x_0) - k$ and let $\delta > 0$ be such that (2.2) holds. Then for $||x - x_0|| < \delta$, $f(x) > f(x_0) - \epsilon = k$. So $\{x : ||x - x_0|| < \delta\} \subset f^{-1}((k, \infty)$.

Conversely, suppose that for every k, $f^{-1}((k, \infty)$ is open for every $k \in \mathbb{R}$. In particular for $\epsilon > 0$ and $x_0 \in X$, the set $f^{-1}((f(x_0) - \epsilon, \infty) \ni x_0$ is open. Hence there exists $\delta > 0$ such that for $||x - x_0|| < \delta$, $x \in f^{-1}((f(x_0) - \epsilon, \infty)$. Therefore $f(x) > f(x_0) - \epsilon$. ☐

The following is a very important result which is a direct consequence of Theorem 2.3

Theorem 2.4. *Let* $\{f_\alpha, \quad \alpha \in A\}$ *be a collection of lower semicontinuous functions. Then*

$$f(x) = \sup_{\alpha \in A} f_\alpha(x) \quad \textit{is lower semicontinuous}$$

Similarly, if $\{f_\alpha, \quad \alpha \in A\}$ *be a collection of upper semicontinuous functions. Then*

$$f(x) = \inf_{\alpha \in A} f_\alpha(x) \quad \textit{is upper semicontinuous}$$

Proof. One just needs to show that

$$\text{if} \quad f(x) = \sup_{\alpha \in A} f_\alpha(x) \quad \text{then} \quad f^{-1}((k, \infty)) = \bigcup_\alpha f_\alpha^{-1}((k, \infty)),$$

$$\text{if} \quad f(x) = \inf_{\alpha \in A} f_\alpha(x) \quad \text{then} \quad f^{-1}((-\infty, k)) = \bigcup_\alpha f_\alpha^{-1}((-\infty, k)).$$

Let us consider the first statement. If $f = \sup_{\alpha \in A} f_\alpha$ and $x \in f^{-1}(k, \infty)$, then and $f(x) > k$. This implies that $f_\alpha(x) > k$ for some α. Otherwise $f(x) \leq k$. Therefore $x \in \bigcup_\alpha f_\alpha^{-1}((k, \infty))$. Reciprocally, if $x \in \bigcup_\alpha f_\alpha^{-1}((k, \infty))$, then $f_\alpha(x) > k$ for some α. In this case $f(x) > k$ and hence $x \in f^{-1}(k, \infty)$. ☐

Next we study the notion of oscillation of a function.

Definition 2.6. *Let* $X \subset \mathbb{R}^n$, *and let* $f : X \longrightarrow \mathbb{R}$ *be a bounded function. The oscillation of* f *over* X *is defined to be:*

$$\omega(f, X) = \sup\{|f(x) - f(y)|; \quad x, y \in X\}.$$

Exercise 2.3. Let $X \subset \mathbb{R}^n$ and let $f : X \longrightarrow \mathbb{R}$ be a bounded function on $X \subset \mathbb{R}^n$, show that

$$\omega(f, X) = \sup_{x \in X} f(x) - \inf_{x \in X} f(x).$$

Next we localize to a point the notion of oscillation of a function.

Definition 2.7. *Let* $X \subset \mathbb{R}^n$ *be an open subset, let* $f : X \longrightarrow \mathbb{R}$ *be a bounded function, and let* $x_0 \in [a, b]$. *Let* $B(x_0, \delta) = \{y \in \mathbb{R}^n : ||y - x_0|| < \delta\}$. *We define the oscillation of* f *at* x_0 *as*

$$\omega(f, x_0) = \lim_{\delta \to 0} \omega(f, B(x_0, \delta)).$$

Notice that if $\delta_1 < \delta_2$ then $\omega(f, B(x_0, \delta_1)) \leq \omega(f, B(x_0, \delta_2))$, so $\omega(\delta) = \omega(f, B(x_0, \delta))$ is a bounded non-decreasing function. Hence the limit exists.

Theorem 2.5. *Let $X \subset \mathbb{R}^n$ be open, and let $f : X \longrightarrow \mathbb{R}$ be a bounded function and let $x_0 \in X$. Then f is continuous at x_0 if and only if $\omega(f, x_0) = 0$.*

Proof. If f is continuous at x_0, then for every $\epsilon > 0$ there exists $\delta > 0$ such that $||x - x_0|| < \delta$ implies $f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$. This says that

$$\sup\{f(x), x \in B(x_0, \delta)\} < f(x_0) + \epsilon, \quad \text{and} \quad \inf\{f(x), x \in B(x_0, \delta)\} > f(x_0) - \epsilon.$$

Thus

$$\omega(f, B(x_0, \delta)) < 2\epsilon.$$

Hence $\omega(f, x_0) = 0$.

Conversely, suppose that $\omega(f, x_0) = 0$. Then

$$\lim_{\delta \to 0} \omega(f, B(x_0, \delta)) = 0.$$

That is, for $\epsilon > 0$ there exists $\delta_0 > 0$ such that for $\delta < \delta_0$ $\omega(f, B(x_0, \delta)) < \epsilon$. But $\omega(f, B(x_0, \delta)) = \sup\{|f(x) - f(y)|, x, y \in B(x_0, \delta)\}$. In particular, for $||x - x_0|| < \delta$, we have $|f(x) - f(x_0)| < \epsilon$. ☐

Theorem 2.6. *Let $X \subset \mathbb{R}^n$ be open and let $f : X \longrightarrow \mathbb{R}$ be bounded and let $x_0 \in X$. The function $\omega(f, x)$ is upper semicontinuous on X. That is fixed $x_0 \in X$ then for all $\epsilon > 0$ there exists $\delta > 0$ such that $x \in X$ and $|x - x_0| < \delta$ implies that $\omega(f, x) < \omega(f, x_0) + \epsilon$.*

Proof. Since $\omega(f, x_0) = \lim_{\delta \to 0} \omega(f, B(x_0, \delta))$, it follows that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\omega(f, B(x_0, \delta)) < \omega(f, x_0) + \epsilon.$$

On the other hand, recalling that if $\delta_1 < \delta_2$ then $\omega(f, B(x_0, \delta_1)) \leq \omega(f, B(x_0, \delta_2))$, $\omega(f, x) < \omega(f, B(x_0, \delta))$, $x \in B(x_0, \delta)$. This proves the theorem. ☐

We deduce from Theorem 2.6 and Theorem 2.3 that

Corollary 2.1. *Let $X \subset \mathbb{R}^n$ be open and let $f : X \longrightarrow \mathbb{R}$ be bounded. Then for every $\alpha > 0$, the set $E_\alpha = \{x \in [a, b] : \omega(f, x) \geq \alpha\}$ is closed.*

Theorem 2.7. *Let $X \subset \mathbb{R}^n$ be open and let $f : X \longrightarrow \mathbb{R}$ be bounded. Let D be the set of points of $[a, b]$ where f is not continuous. For $\delta > 0$, let $E_\delta = \{x \in [a, b] : \omega(f, x) \geq \delta\}$. Then*

$$D = \bigcup_{\delta > 0} E_\delta = \bigcup_{n \in \mathbb{N}} E_{1/n}.$$

Proof. We know that f is not continuous at x_0 if and only if $\omega(f, x_0) > 0$. But it is easy to see that

$$\{x, \omega(f, x) > 0\} = \bigcup_{\alpha > 0} \{x : \omega(f, x) \geq \alpha\}.$$

Since $E_\delta \subset E_\gamma$, for $\delta > \gamma$, $D = \bigcup_{n \in \mathbb{N}} E_{1/n}$. ☐

The structure of the set D is important and deserves a name:

Definition 2.8. *A set $F \subset \mathbb{R}^n$ is an F_σ if*

$$F = \bigcup_{n=1}^{\infty} F_n, \quad F_n \quad \textit{closed} .$$

A set $G \subset \mathbb{R}^n$ *is a* G_δ *if*

$$G = \bigcap_{n=1}^{\infty} G_n, \quad G_n \quad \textit{open.}$$

So, theorem 2.7 implies that, D is an F_σ.

3. Sets of Measure Zero and Cantor Sets

We want to make sense of a way to characterize when a subset $X \subset \mathbb{R}^n$ is small. For now we will restrict ourselves to one dimension.

Definition 3.1. *A subset* $X \subset \mathbb{R}$ *has content zero, and write* $c(X) = 0$, *if for an arbitrary* $\epsilon > 0$, *there exist a finite collection of open intervals* $I_1, I_2, ..., I_N$ *such that*

$$X \subset \bigcup_{j=1}^{N} I_j, \quad \textit{and} \quad |I_1| + |I_2| + ... + |I_N| < \epsilon.$$

Here $|I_j|$ *denotes the length of the interval* I_j.

We say that X *has Lebesgue measure equal to zero, and denote* $m(X) = 0$, *if for every* $\epsilon > 0$ *there is a countable family of open intervals* $\{I_j, \ \ j \in \mathbb{N}\}$, *such that*

$$X \subset \bigcup_{j=1}^{\infty} I_j, \quad \textit{and} \quad \sum_{j=1}^{\infty} |I_j| < \epsilon.$$

In particular, if $c(X) = 0$, *then* $m(X) = 0$.

Exercise 3.1. Show that one can take the intervals to be closed in the definition 3.1.

Theorem 3.1. *Sets of Lebesgue measure zero have the following properties*
1) If $m(X) = 0$ *and* $Y \subset X$ *then* $m(Y) = 0$.
2) If X *is compact and* $m(X) = 0$, *then* $c(X) = 0$.
3) If $Y = \bigcup_{j=1}^{\infty} X_j$ *and* $m(X_j) = 0$, *for all* $j \in \mathbb{N}$, *then* $m(Y) = 0$.

We leave the proof as an exercise.

Next we give examples of some "weird" sets, which will be a rich source of examples and counter-examples during the course, and therefore it is useful to review them.

Let C be the subset of $[0, 1]$ defined in the following way: Let $r \in (0, 1)$. Let $E_0 = [0, 1]$. Let E_1 be the set obtained by removing the open interval of length r centered at $\frac{1}{2}$. That is:

$$E_1 = [0, \frac{1}{2}(1-r)] \cup [\frac{1}{2}(1+r), 1].$$

So E_1 consists of two intervals of length $\frac{1}{2}(1-r)$. Therefore the sum of the lengths of the intervals that make-up E_1 is $1-r$. These can be thought of two copies of $[0, 1]$scaled by the factor $\frac{1}{2}(1-r)$. Then repeat the process, taking into account the scaling factor. So let E_2 be the closed set obtained from E_1 by removing the open interval centered at the middle of each interval of E_1, of length $\frac{1}{2}(1-r)r$ and so on. So E_2 will consist of four intervals of length $\frac{1}{4}(1-r)^2$.

In the general case, if $n > 1$, the set E_n is obtained from E_{n-1} by removing 2^{n-1} open intervals of length $\left(\frac{1}{2}(1-r)\right)^{n-1} r$. So E_n consists of 2^n intervals of length $\left(\frac{1}{2}(1-r)\right)^n$. Notice that the sum of the lengths of the intervals that make up the set E_n is $(1-r)^n$.

So we have

$$E_0 \supset E_1 \supset E_2 \supset ... \supset E_n$$

The Cantor set is defined by

$$C = \bigcap_{n=1}^{\infty} E_n.$$

Theorem 3.2. *C is compact and its content is equal to zero.*

Proof. C is the intersection of closed sets, so it is closed. Since it is contained in $[0, 1]$ it must be compact.

Given $\delta > 0$, pick N such that $(1 - r)^N < \delta/2$. Since $E_N \supset E$, one can cover E by finitely many intervals with sum of their lengths less than δ. □

Now we will construct a similar set, but with measure not equal to zero.

As before, let $F_0 = [0, 1]$. Let F_1 be the set obtained by removing the middle open interval of length r. That is:

$$F_1 = [0, \frac{1}{2}(1 - r)] \cup [\frac{1}{2}(1 + r), 1].$$

So the sum of the lengths of the intervals that make-up E_1 is $1 - r$. Now, instead of removing the interval of length $\frac{1}{2}(1 - r)r$ from each interval, we remove less. Let's say, we remove $\frac{1}{2}(1 - r)r^2$ from F_1. So let F_2 be the closed set obtained from F_1 by removing the middle open interval of length $\frac{1}{2}(1 - r)r^2$ of each interval of F_1. So F_2 consists of four intervals of length $l_2 = \frac{1}{4}(1 - r)(1 - r^2)$ each.

Let F_3 be the set obtained from F_2 by removing the middle open interval of length $l_2 r^3$ from each interval of F_2. So F_3 consists of 8 intervals of length $2^{-3}(1 - r)(1 - r^2)(1 - r^3)$. In general, if we let l_n denote the length of each interval of F_n, then F_{n+1} is the set obtained by removing the middle open interval of length $l_n r^{n+1}$ from F_n.

Lemma 3.1. *For $n \geq 1$, F_n consists of 2^n intervals of length*

$$l_n = 2^{-n} \prod_{j=2}^{n} \left(1 - r^j\right). \tag{3.1}$$

Proof. By definition $F_1 = [0, \frac{1}{2}(1 - r)] \cup [\frac{1}{2}(1 + r), 1]$. So the claim holds for $n = 1$.

Supposes (3.1) holds for n holds and we want to show it holds for $n + 1$.

Since we have 2^{n+1} intervals of the same size, we work with the one which contains 0.Take the interval $[0, l_n] = l_n[0, 1]$ and remove its middle open interval of length $l_n r^{n+1}$. So we get two intervals of the same length, and the one which contains zero is $l_n \left[0, \frac{1}{2}(1 - r^{n+1}\right] = [0, l_n \frac{1}{2}(1 - r^{n+1}$. But $l_{n+1} = l_n \frac{1}{2}\left(1 - r^{n+1}\right)$. This ends the proof of the lemma □

As above, the modified Cantor set is defined by

$$K = \bigcap_{n=1}^{\infty} F_n.$$

Unlike the first Cantor set C, the sums of the lengths of intervals of F_n do not go to zero as $n \to \infty$. In this case the sum of the intervals that make-up F_n is

$$M_n = 2^n L_n = \prod_{j=1}^{n} \left(1 - r^j\right)$$

But we know that

$$\prod_{j=1}^{\infty} \left(1 - r^{j}\right) \neq 0$$

Theorem 3.3. *The Cantor set K is compact, has empty interior, but it does not have measure zero.*

Proof. K is closed and contained in $[0,1]$. Therefore it is compact.

Since F_n consists of 2^n disjoint intervals of length $l_n = 2^{-n}\prod_{j=1}^{n}(1-r^j)$, It is easy to see that K has empty interior. Otherwise K, and hence F_n, $n = 1, 2, ...$, would contain an interval I of size δ. Since the intervals that make-up F_n are disjoint, I would have to be contained entirely inside one of its intervals. Just pick n so that $l_n < \delta$.

If K had measure zero, then for any $\epsilon > 0$ there would be a family I_n, $n \in \mathbb{N}$, of open intervals such that $K \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \epsilon$. Since K is compact, this collection of intervals can be assumed to be finite, so K would have content zero.

Then

$$K \subset I_1 \cup I_2 \cup ... \cup I_N, \quad \sum_{j=1}^{N} |I_j| < \epsilon.$$

Pick $\epsilon < \prod_{j=1}^{\infty}(1-r^j)$. Let $U = I_1 \cup I_2 \cup ... \cup I_N$. We claim that there exists $m \in \mathbb{N}$ such that $U \supset F_m$, and hence $U \supset F_j$ for all $j \geq m$. Suppose this is not true. That is for all $m \in \mathbb{N}$, $G_m = F_m \setminus U \neq \emptyset$. Since U is open, G_m is closed, and therefore compact, and $G_j \supset G_{j+1}$. By theorem 2.2

$$\bigcap_{m\in\mathbb{N}} G_m = \left(\bigcap_{m\in\mathbb{N}} F_m\right) \setminus U \neq \emptyset.$$

That is absurd.

So $U \supset F_m$ for some m. Therefore ϵ would have to be bigger than or equal to the sum of the lengths of the intervals of F_m. That is $\epsilon \geq \prod_{j=1}^{m}(1-r^j) > \prod_{j=1}^{m}(1-r^j)$. That is not possible, in view of the choice of ϵ.

□

4. The Riemann Integral in $\mathbb{R}$.

In this section we will discuss the Riemann integral. For simplicity we restrict ourselves to $\mathbb{R}$, but all the results proved here can be easily extended to higher dimensions. I believe most students in this class have studied this before. Our goal is to do a brief review of the methods, and to emphasize their limitations.

A partition $\mathcal{P}$ of an interval $[a,b]$, $a, b < \infty$, is a finite collection of points $\{x_0, x_1, ..., x_n\} \subset [a,b]$ with $a = x_0$, $x_j \leq x_{j+1}$, and $b = x_n$. We say that a partition $\mathcal{P}_1$ is a refinement of $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{P}_1$.

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function, and let $\mathcal{P} = \{x_1, x_2, ..., x_n\}$ be a partition of $[a,b]$.

We define the upper and lower sums with respect to a fixed partition $\mathcal{P}$.

$$U(\mathcal{P}, f) = \sum_{j=1}^{n} M_j \left(x_{j+1} - x_j\right), \quad M_j = \sup_{[x_1, x_{j+1}]} f(x),$$

$$L(\mathcal{P}, f, \alpha) = \sum_{j=1}^{n} m_j \left(x_{j+1} - x_j\right), \quad m_j = \inf_{[x_1, x_{j+1}]} f(x).$$

Obviously,

$$L(\mathcal{P}, f, \alpha) \leq U(\mathcal{P}, f, \alpha).$$

Proposition 4.1. *Let* $\mathcal{P}$ *and* $\mathcal{P}_1$ *be partitions of* $[a,b]$*. Suppose* $\mathcal{P} \subset \mathcal{P}_1$*, i.e.* $\mathcal{P}_1$ *is a refinement of* $\mathcal{P}$*. Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be bounded and let* $M = \sup_{[a,b]} f(x)$*, and* $m = \inf_{[a,b]} f(x)$*. Then*

$$m(b-a) \leq L(\mathcal{P}, f) \leq L(\mathcal{P}_1, f) \leq U(\mathcal{P}_1, f) \leq U(\mathcal{P}, f) \leq M(b-a).$$

Proof. Since $M_j \leq M$ and $m_j \geq m$,

$$U(\mathcal{P}, f) \leq M(b-a) \quad \text{and} \quad m(b-a) \leq L(\mathcal{P}, f).$$

We use induction to prove the other inequalities. Suppose that $\mathcal{P}_1$ contains one more point than $\mathcal{P}$. Let $\{x^*\} = \mathcal{P}_1 \setminus \mathcal{P}$ and suppose $x^* \in [x_j, x_{j+1}]$. Let

$$M_{j,1} = \sup_{[x_j, x^*]} f(x), \quad M_{j,2} = \sup_{[x^*, x_{j+1}]} f(x).$$
$$m_{j,1} = \inf_{[x_j, x^*]} f(x), \quad m_{j,2} = \inf_{[x^*, x_{j+1}]} f(x).$$

Recall that $M_j = sup_{[x_j, x_{x+1}]} f(x)$ and $m_j = inf_{[x_j, x_{j+1}]} f(x)$. It is then clear that

$$M_{j,1} \leq M_j, \quad M_{j,2} \leq M_j,$$
$$m_{j,1} \geq m_j, \quad m_{j,2} \geq m_j.$$

Then

$$U(\mathcal{P}_1, f) - U(\mathcal{P}, f) = M_{j,1}(x^* - x_j) + M_{j,2}(x_{j+1} - x^*) - M_j(x_{j+1} - x_j) \leq 0.$$
$$L(\mathcal{P}_1, f) - L(\mathcal{P}, f) = m_{j,1}(x^* - x_j) + m_{j,2}(x_{j+1} - x^*) - m_j(x_{j+1} - x_j) \geq 0.$$

If $\mathcal{P}_1 \setminus \mathcal{P}$ consists of k points, one just needs to repeat this argument k times. □

Definition 4.1. *The upper and lower integrals of* f *in* $[a,b]$ *are respectively*

$$\overline{\int_a^b} f \, dx = \inf_{\mathcal{P}} U(\mathcal{P}, f), \quad \underline{\int_a^b} f \, dx = \sup_{\mathcal{P}} L(\mathcal{P}, f).$$

Definition 4.2. *We say that* f *is integrable, and denote* $f \in \mathcal{R}$*, if*

$$\overline{\int_a^b} f \, dx = \underline{\int_a^b} f \, dx \overset{\text{def}}{=} \int_a^b f \, dx.$$

It is very important to characterize the class of functions which are integrable in the sense of Riemann.

The concept of oscillation of a function over a set is an important one, not just for the theory of integration.

Theorem 4.1. *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be a bounded function. The following are equivalent:*
1) $f \in \mathcal{R}$*.*
2) For every $\epsilon > 0$ *there exist partitions* $\mathcal{P}$ *and* $\mathcal{Q}$ *of* $[a,b]$ *such that* $U(f, \mathcal{P}) - L(f, \mathcal{Q}) < \epsilon$*.*
3)For every $\epsilon > 0$ *there exists a partition* $\mathcal{P}$ *of* $[a,b]$ *such that* $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$*.*
4) For every $\epsilon > 0$ *there exists a partition* $\mathcal{P} = \{x_0, ..., x_n\}$ *of* $[a,b]$ *such that*

$$\sum_{j=1}^{n} \omega_j (x_{j+1} - x_j) < \epsilon, \quad \omega_j = \omega(f, [x_j, x_{j+1}]).$$

Proof. The implications $1 \Longleftrightarrow 2$ follow almost directly from the defintion. It is obvious that $3 \Longrightarrow 2$, just take $\mathcal{P} = \mathcal{Q}$. If 2 holds, let $\mathcal{P}_1 = \mathcal{P} \cup \mathcal{Q}$. From Proposition 4.1,

$$L(f, \mathcal{Q}) \leq L(f, \mathcal{P}_1) \leq U(f, \mathcal{P}_1) \leq U(f, \mathcal{P}).$$

Then $U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1) < \epsilon$.

The implications $4 \Longleftrightarrow 3$ follow from exercise 2.3. □

Corollary 4.1. *If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then $f \in \mathcal{R}$.*

Proof. Since $[a,b]$ is compact, and f is continuous, f is bounded. The compactness of $[a,b]$ also implies that f is uniformly continuous. So for any $\epsilon > 0$, we can pick $\delta > 0$ such that for any $x, y \in [a,b]$ with $|x-y| < \delta$, $|f(x) - f(y)| < \epsilon/(b-a)$.

Let $\mathcal{P} = \{x_0, ..., x_n\}$ be a partition of $[a,b]$ such that $|x_j - x_{j+1}| < \delta$. Hence $\omega_j(f) = \omega(f, [x_j, x_{j+1}]) < \epsilon/(b-a)$. Then

$$\sum_{j=1}^{n} \omega_j (x_{j+1} - x_j) < \epsilon/(b-a) \sum_{j=1}^{n} (x_{j+1} - x_j) = \epsilon.$$

In view of Theorem 4.1, $f \in \mathcal{R}$. □

The goal of this section is to prove

Theorem 4.2. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function and let $D \subset [a,b]$ be the set of points where f is not continuous. Then $f \in \mathcal{R}$ if and only if $m(D) = 0$.*

In view of Theorem 3.1, Corollary 2.1 and Theorem 2.7, Theorem 4.2. follows directly from

Theorem 4.3. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}$ if and only if for every $\delta > 0$ the set $E_\delta = \{x \in [a,b] : \omega(f,x) \geq \delta\}$ has content equal to zero.*

Indeed, suppose Theorem 4.3 has been proved. Suppose first that $m(D) = 0$. Since $D = \bigcup_{n \in \mathbb{N}} E_{1/n}$, we have $m(E_{1/n}) = 0$, for all $n \in \mathbb{N}$. Since $E_\delta \subset E_\gamma$, for $\delta > \gamma$, it follows from Theorem 3.1 that $m(E_\delta) = 0$ for all $\delta > 0$. Since E_δ is compact, by Corollary 2.1, Theorem 3.1 guarantees that $c(E_\delta) = 0$.

Conversely, if $c(E_\delta) = 0$ for all $\delta > 0$, then in particular $c(E_{1/n}) = m(E_{1/n}) = 0$ for all $n \in \mathbb{N}$. Then Theorem 3.1 implies that $m(D) = 0$.

Now we prove Theorem 4.3.

Proof. The implication $f \in \mathcal{R} \Longrightarrow c(E_\delta) = 0$, for all $\delta > 0$, is easier and we prove it first.

Suppose that $f \in \mathcal{R}$ and fix $\delta > 0$. We want to show that $c(E_\delta) = 0$. Let $\epsilon > 0$. Since $f \in \mathcal{R}$, we know that there exists a partition $\mathcal{P} = \{x_0, ..., x_n\}$ of $[a,b]$ such that

$$\sum_{j=1}^{n} \omega_j (x_{j+1} - x_j) < \epsilon\delta.$$

Let $I_j = [x_j, x_{j+1}]$ and let $A = \{j : 1 \leq j \leq n, \text{ and } I'_j \cap E_\delta \neq \emptyset\}$. Since for every $x \in E_\delta \cap I_k$, $\delta \leq \omega(f,x) \leq \omega(f, I_k)$, it follows that

$$\delta \sum_{j \in A} |I_j| = \delta \sum_{j \in A} (x_{j+1} - x_j) \leq \sum_{j \in A} \omega_j (x_{j+1} - x_j) \leq \sum_{j=1}^{n} \omega_j (x_{j+1} - x_j) < \epsilon\delta.$$

Hence

$$\sum_{j \in A} |I_j| < \epsilon.$$

The intervals I_j cover E_δ, it follows that $c(E_\delta) = 0$.

Now we prove the converse. We begin with the following

Lemma 4.1. *Suppose that $\omega(f,x) < \delta$ for all $x \in [p,q]$. Then there exists a partition $\mathcal{P} = \{x_0, x_1, ..., x_N\}$ of $[p,q]$ such that*

$$M_j - m_j < \delta, \quad j = 1, 2, ..., N, \quad M_j = \sup_{[x_j, x_{j+1}]} f(x), \quad m_j = \inf_{[x_j, x_{j+1}]} f(x). \tag{4.1}$$

Proof. Extend $f(x) = f(p)$ for $x < p$ and $f(x) = f(q)$ for $x > q$. For each $x \in [p, q]$ there exists an open interval $I_x \ni x$ such that $\sup_{x \in \overline{I_x}} f(x) - \inf_{x \in \overline{I_x}} f(x) < \delta$, where $\overline{I_x}$ is the closure of I_x. This gives an open cover of $[p, q]$ and hence it has a finite subcover $I_{x_1}, ..., I_{x_N}$. Now take a partition $\mathcal{P}$ of $[p, q]$ such that an interval of $\mathcal{P}$ is contained in the closure of one of the intervals I_{x_j}. Then (4.1) holds. □

Suppose $c(E_\delta) = 0$, for all δ. We want to show that $f \in \mathcal{R}$.

Let $\delta > 0$. Since $c(E_\delta) = 0$, there exist open intervals $I_1, ..., I_N$ such that $E_\delta \subset \bigcup_{j=1}^N I_j$, and that $|I_1| + ... + |I_N| < \delta$. We begin with a partition $\mathcal{P}$ of $[a, b]$ consisting of intervals that fall in two categories $\mathcal{P}_1$ and $\mathcal{P}_2$ defined as follows: $J \in \mathcal{P}_1$ if $IJ \subset \overline{I_j}$ for some j, and $J \in \mathcal{P}_2$ if $J \cap E_\delta = \emptyset$.

Let J be an interval in $\mathcal{P}_2$. Since $\omega(f, x) < \delta$ for all $x \in J$, Lemma 4.1 shows that there exists a partition $\mathcal{P}_J$ of J such that $\sup_J f(x) - \inf_{x \in J} f(x) < \delta$. Doing this for every interval in $\mathcal{P}_2$ we get a refinement $\mathcal{P}'$ of $\mathcal{P}$ such that each interval of $\mathcal{P}'$ falls in two classes, defined as above, which we denote by $\mathcal{P}'_1$ and $\mathcal{P}'_2$.

Hence

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = \sum_{J \in \mathcal{P}'_1} (M_J - m_J)|J| + \sum_{J \in \mathcal{P}'_2} (M_J - m_J)|J|, \quad M_j = \sup_J f(x), \quad m_j = \sup_J f(x).$$

Since f is bounded $|f| \leq M$ and hence $M_J - m_J \leq 2M$ for any $J \in \mathcal{P}'$.

For $J \in \mathcal{P}'_2$, $M_J - m_J < \delta$. Hence

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = \sum_{J \in \mathcal{P}_1} 2M|J| + \sum_{J \in \mathcal{P}_2} \delta|J| < (2M + b - a)\delta.$$

Since $\delta > 0$ is arbitrary, this shows that $f \in \mathcal{R}$. □

4.1. **Limitations of the Riemann Integral.** Theorem 4.2 shows that the class of functions integrable in the sense of Riemann is small. The fisrt limitations is that one can have two functions which are equal outside a set of measure zero, with one integrable and the other not integrable.

Consider the following function

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$

Exercise 4.1. Show that $\omega(\chi_{\mathbb{Q}}, x) = 1$.

Therefore, $\chi_{\mathbb{Q}}$ is discontinuous everywhere, and hence not Riemann integrable. On the other hand $F(x) = 0$ is Riemann integrable, and $F(x) = \chi_{\mathbb{Q}}(x)$, $x \notin \mathbb{Q}$. So they are equal outside a set of measure zero, one is integrable and the other one is not.

Proposition 4.2. *Let χ_C and χ_K be the characteristic functions of the Cantor sets defined above. Then the set of discontinuities of χ_C and χ_K are C and K respectively.*

Proof. Let us take the case of χ_C. The other one is identical. Since C is closed, $\mathbb{R} \setminus C$ is open. Since $\chi_C(x) = 0$ if $x \notin C$, χ_C is continuous in $\mathbb{R} \setminus C$. Since C has empty interior, for every point of C there is a sequence of points of $\mathbb{R} \setminus C$ converging to it. Hence the set of discontinuities of χ_C is exactly equal to C. □

Corollary 4.2. $\chi_C \in \mathcal{R}$, *but* $\chi_K \notin \mathcal{R}$.

Another limitation of the Riemann integral is in its relation with limit operations. Before we discuss these examples, we will review some concepts about convergence of functions in the next section.

5. Metric Spaces

Several, or perhaps all, topics discussed in this section are not new for most students in MA 544. However these concepts are an important part of the course, and worth reviewing.

Definition 5.1. *A metric space is a pair (M, d) consisting of a set M and a function*

$$d : M \times M \longrightarrow [0, \infty)$$

satisfying the following properties:
1) $d(x, y) = d(y, x)$ for all $x, y \in M$
2) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in M$ (triangle inequality)
3) $d(x, y) = 0$ if and only if $x = y$.
In what follows we denote a metric space either by M, or (M, d), if we want to emphasize its metric.

Example 1) $M = \mathbb{R}^n$, $n \geq 1$ and $d(x, y) = \left((x_1 - y_1)^2 + ... + (x_n - y_n)^2\right)^{\frac{1}{2}}$.

Exercise 5.1. Verify that $(\mathbb{R}^n, d)$ is a metric space.

Example 2) Let $C([0, 1]) = M$ denote the set of real valued functions defined in $[0, 1]$. Let

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Properties 1 and 3 are obvious. One just needs to verify property 2. Let $f, g, h \in C([0, 1])$, Since by the standard triangle inequality

$$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|,$$

we immediately see that

$$d(f, h) \leq d(f, g) + d(g, h).$$

Definition 5.2. *Let X be a vector space over $\mathbb{R}$ or m, A norm on X is a function*

$$N : X \longrightarrow [0, \infty)$$

such that

$$\begin{gathered} N(x) = 0 \Longleftrightarrow x = 0 \\ N(\lambda x) = |\lambda| N(x), \quad \lambda \in \mathbb{R} \ \textit{or} \quad \mathbb{C} \\ N(x + y) \leq N(x) + N(y). \end{gathered}$$

One often uses the notation $N(x) = ||x||$.

Exercise 5.2. Let (X, N) be a normed vector space. Show that

$$d(x, y) = N(x - y)$$

is a metric on X.

5.1. **Sequences and their convergence.** A sequence of elements of a metric space M is a countable subset $\{x_j, \ \ j \in \mathbb{N}\}$ of elements $x_j \in M$.

Definition 5.3. *A sequence* $\{x_j\}$ *converges to* $x \in M$ *if for every* $\epsilon > 0$, *there exists* $N \in \mathbb{N}$, *which depends on* ϵ, *such that for all* $j > N$, $d(x_j, x) < \epsilon$. *In this case we say that*

$$\lim_{j \to \infty} x_j = x \quad \textit{or} \quad x_j \to x.$$

An important concept is that of a Cauchy sequence.

Definition 5.4. *A sequence* $\{x_j\}$ *is Cauchy if for every* $\epsilon > 0$, *there exists* $N \in \mathbb{N}$, *which depends on* ϵ, *such that for all* $j, k > N$, $d(x_j, x_k) < \epsilon$.

Theorem 5.1. *Every convergent sequence is Cauchy.*

Proof. This is an easy consequence of the triangle inequality. Let $x_j \in M$, with $x_j \to x$. For $\epsilon > 0$, pick N such that $d(x_j, x) < \epsilon/2$, for all $j > N$. Then for $j, k > N$,

$$d(x_j, x_k) \leq d(x_j, x) + d(x_k, x) < \epsilon.$$

Therefore x_j is Cauchy. $\square$

It is not the case that all Cauchy sequences in a metric space (M, d) converge to an element $x \in M$.

Example 5.1. Let $M = \mathbb{Q}$ denote the set of rational numbers with the metric $d(p, q) = |p - q|$. $\mathbb{Q}$ is not complete.

To see that, pick a sequence $q_j \in \mathbb{Q}$ which converges to $\sqrt{2}$. $\{q_j\}$ is Cauchy, as it converges in $\mathbb{R}$, but it does not converge in $\mathbb{Q}$.

Definition 5.5. *A metric space* (M, d) *is complete if for every Cauchy sequence* $\{x_j\} \subset M$ *there exists* $x \in M$ *such that* $\{x_j\}$ *converges to* x.

5.2. **Completion of metric spaces.** One wonders if given a metric space, there is a complete metric space that contains it.

Definition 5.6. *Let* (M, d) *and* $(\widetilde{M}, \widetilde{d})$ *be two metric spaces. A map*

$$i : M \longrightarrow \widetilde{M}$$

is an isometry if

$$\widetilde{d}(i(x), i(y)) = d(x, y)$$

(Notice that i *is injective). In this case we call* i *an embedding of* M *into* $\widetilde{M}$. *If* i *is onto, we say that* M *and* $\widetilde{M}$ *are isomorphic.*

Theorem 5.2. *Let* (M, d) *be a metric space. Then there is a metric space,* $(\widetilde{M}, \widetilde{d})$ *with the following properties:*
1) $(\widetilde{M}, \widetilde{d})$ *is complete,*
2) There is an embedding $i : M \longrightarrow \widetilde{M}$,
3) $i(M)$ *is dense in* $\widetilde{M}$.
4) If (X, ρ) *is another metric space satisfying 1,2 and 3, then* $(\widetilde{M}, \widetilde{d})$ *and* (X, ρ) *are isomorphic.*

Proof. Will be in the homework assignment. $\square$

6. Spaces of Functions

The most important examples of metric spaces in MA 544 are certain spaces of functions. We begin with the following fundamental examples, which are the motivation for the introduction of the Lebesgue measure.

Exercise 6.1. Let $M = C([0,1])$ be the space of continuous real valued functions in $[0,1]$. Let

$$||f||_\infty = \sup_{x\in[0,1]} |f(x)|,$$

$$||f||_p = \left[\int_0^1 |f(x)|^p dx\right]^{1/p}, \quad p \geq 1. \quad \text{This is the Riemann integral.}$$

Show that $||f||_p$, $1 \leq p \leq \infty$ are norms.

Theorem 6.1. *The metric space* $(C([0,1], d_p)$, $d_p(f,g) = ||f-g||_p$ *is incomplete if* $1 \leq p < \infty$ *and is complete if* $p = \infty$.

Proof. It is easier to prove that $(C([0,1], d_p)$, $1 \leq p < \infty$, is incomplete. Let

$$f_n(x) = \begin{cases} 1 \quad \text{if} \quad x \in [0, \frac{1}{2}] \\ \\ 1 - \frac{n}{2}(x - \frac{1}{2}) \quad \text{if} \quad x \in [\frac{1}{2}, \frac{1}{2} + \frac{2}{n}] \\ \\ 0 \quad \text{if} \quad x \in [\frac{1}{2} + \frac{2}{n}, 1] \end{cases}$$

Let

$$f(x) = \begin{cases} 1 \quad \text{if} \quad x \in [0, \frac{1}{2}] \\ \\ 0 \quad \text{if} \quad x \in [\frac{1}{2}, 1] \end{cases}$$

Then

$$f(x) - f_n(x) = \begin{cases} 0 \quad \text{if} \quad x \in [0, \frac{1}{2}] \\ \\ 1 - \frac{n}{2}(x - \frac{1}{2}) \quad \text{if} \quad x \in [\frac{1}{2}, \frac{1}{2} + \frac{2}{n}] \\ \\ 0 \quad \text{if} \quad x \in [\frac{1}{2} + \frac{2}{n}, 1] \end{cases}$$

Hence

$$\int_0^1 |f(x) - g(x)|^p \, dx = \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{2}{n}} (1 - \frac{n}{2}(x - \frac{1}{2}))^p dx \leq \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{2}{n}} dx = \frac{2}{n}.$$

So $d_p(f_n, f) \to 0$, $1 \leq p < \infty$, as $n \to \infty$. But $f \notin C([0,1])$.

Now we prove that the metric space $(C([0,1], d_\infty)$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $(C([0,1], d_\infty)$. Then for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$||f_n(x) - f_m(x)||_\infty < \epsilon \quad \text{for all} \quad m, n \in \mathbb{N}. \tag{6.1}$$

Fix $x \in [0,1]$. Then $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$, and therefore it converges. Let

$$f(x) \overset{\text{def}}{=} \lim_{n\to\infty} f_n(x).$$

In view of (6.1) f_n converges to f uniformly on $[0,1]$. Since f_n is continuous and $[0,1]$ is compact, $f \in C([0,1])$. This ends the proof of the Theorem. □

We know from Theorem 5.2 that $(C([0,1]), d_p)$, $1 \leq p < \infty$, can be completed. The question is what is its completion? This is a very important question in analysis.

Let $f, g \in \mathcal{R}([0,1])$, i.e. f and g are Riemann integrable in $[0,1]$. We will say that f and g are equivalent if there exists a set F with $m(F) = 0$ such that

$$f(x) = g(x), \quad \forall \;\; x \in [0,1] \setminus F.$$

This is an equivalence relation.

Theorem 6.2. *Let $L^p([0,1], \mathcal{R})$ denote the space of equivalence classes of functions which are Riemann integrable and that for each $f \in L^p([0,1], \mathcal{R})$, let*

$$||f||_p = \left[\int_0^1 |f(x)|^p dx\right]^{\frac{1}{p}}.$$

Then $L^p([0,1], \mathcal{R})$ is a normed vector space, but it is not complete.

Proof. We will take $p = 1$, but the general case is the same. Since the set of discontinuities of $|f|$ is contained in the set of discontinuities of f, and f is Riemann integrable, then $|f|$ also is Riemann integrable. Recall that by assumption if $f \in \mathcal{R}$, then f is bounded. So there is no convergence problem with the integral. It is clear that $||f||_p$ does not depend on the choice of the representative of the class of f.

The properties of the norm are clear, with the exception of its non-degeneracy. We need to show that if $f \in \mathcal{R}$ and $||f||_1 = 0$, then $f \equiv 0$. This follows from

Lemma 6.1. *If $g(x) \geq 0$ for all x in $[0,1] \setminus M$, and $\int_0^1 g(x)dx = 0$ then $g(x) = 0$ for all x such that g is continuous at x.*

Proof. If $g(x_0) \neq 0$ and g is continuous at x_0, then there exists $\delta > 0$ such that if $x - x_0| < \delta$, then $g(x) > g(x_0)/2$. This implies that $\int_0^1 g(x)dx > \delta g(x_0)/2 > 0$. □

If $||f||_1 = 0$, then $|f(x) = 0$ for all x in the set of points where $|f(x)|$ is continuous. Since f is Riemann integrable, it follows that $f = 0$ except on a set of measure zero. Thus $f \equiv= 0$.

To prove that $L^1([0,1], \mathcal{R})$ is incomplete we recall the construction of the Cantor set of non-zero measure.

Let $f_n = \chi_{F_n}$ be the characteristic function of the set F_n defined in Lecture 1. f_n is obviously Riemann integrable. Moreover

$$f_n - f_{n+1} = \chi_{F_n \setminus F_{n+1}}(x)$$

By the construction $F_n \setminus F_{n+1}$ consists of 2^n *disjoint* intervals of length $2^{-n} \prod_{j=1}^n (1 - r^j) r^{n+1}$. Therefore

$$\int_0^1 (f_n - f_{n+1}) dx = \prod_{j=1}^n (1 - r^j) r^{n+1} \leq r^{n+1}$$

If $N > 1$ we write

$$|f_n - f_{n+N}| \leq \left[\sum_{j=1}^N |f_{n+j} - f_{n+j-1}|\right] \leq \sum_{j=1}^N |f_{n+j} - f_{n+j-1}|$$

Hence

$$||f_n - f_{n+N}||_1 \leq \sum_{j=1}^N r^{n+j} \leq \frac{r^n}{1-r}.$$

Hence f_n is a Cauchy sequence in $L^1([0,1],\mathcal{R})$. But it obviously converges to the characteristic function of the Cantor set K which is not Riemann integrable. $\square$