1. ABSTRACT INTEGRATION

The main reference for this section is Rudin's Real and Complex Analysis.

The purpose of developing an "abstract theory of integration" is to emphasize the difference between the Riemann and Lebesgue integrals and show that one can define an integral of a real valued function defined on very general sets.

1.1. σ-algebras and measurable sets.

Definition 1.1. *A collection $\mathcal{M}$ of subsets of a set X is called a σ-algebra in X if it has the following properties:*
1) $X \in \mathcal{M}$.
2) if $A \in \mathcal{M}$ then $X \setminus A = A^c \in \mathcal{M}$.
3) If $A_n \in \mathcal{M}$, $n = 1, 2, 3...$ and $A = \bigcup_{n=1}^{\infty} A_n$, then $A \in \mathcal{M}$.

The pair $(X, \mathcal{M})$ is called a measurable space and the elements of $\mathcal{M}$ are called measurable sets.
Examples:
1) The family of all subsets of X is a σ-algebra.

Proposition 1.1. *If $\mathcal{F}$ is any collection of subsets of X, then there exists the smallest σ-algebra $\mathcal{M}_{\mathcal{F}}$ in X such that $\mathcal{F} \subset \mathcal{M}_{\mathcal{F}}$.*

Proof. Let Ω be the family of all σ-algebras $\mathcal{M}$ in X which contain $\mathcal{F}$. The example above shows that this family is not empty. Let

$$\mathcal{M}_{\mathcal{F}} = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}.$$

Obviously, $\mathcal{F} \subset \mathcal{M}_{\mathcal{F}}$. Now we just need to verify that $\mathcal{M}_{\mathcal{F}}$ is a σ-algebra.

Since $X \in \mathcal{M}$ for all $\mathcal{M} \in \Omega$, we deduce that $X \in \mathcal{M}_{\mathcal{F}}$.

If $A_j \in \mathcal{M}_{\mathcal{F}}$, $j = 1, 2, ...$ it follows that If $A_j \in \mathcal{M}$, $j = 1, 2, ...$, for all $\mathcal{M} \in \Omega$. Since each $\mathcal{M}$ is a σ-algebra, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$, and thus If $\bigcup_{n=1}^{\infty} A_j \in \mathcal{M}_{\mathcal{F}}$. Property 2 can be verified in the same way and we leave it as an exercise. ☐

Now consider the case where X is a metric space (or more generally a topological space, if you know what that is). Let $\mathcal{F}$ consist of the family of all open subsets of X. The family $\mathcal{F}$ is certainly not a σ-algebra, but the Proposition 1.1 states that there exists the smallest σ-algebra containing $\mathcal{F}$. So we define

Definition 1.2. *Let X be a metric (topological) space. The smallest σ-algebra that contains all open subsets of X is called the Borel σ-algebra and will be denoted by $\mathcal{B}(X)$.*

1.2. measurable functions.

Now we introduce the concept of a measurable function.

Definition 1.3. *Let $(X, \mathcal{M})$ be a measurable space and let Y be a metric (topological) space. A function $f : X \longrightarrow Y$ is measurable if $f^{-1}(U)$ is measurable for every open set $U \subset Y$.*

Example 1.1. If $E \subset X$ is measurable, χ_E is measurable. Indeed, if $U \subset \mathbb{R}$ is empty then one of the following possibilities hold.

$$\chi_E^{-1}(U) = \emptyset, \quad \text{if} \quad 0 \notin U, \quad 1 \notin U,$$
$$\chi_E^{-1}(U) = X, \quad \text{if} \quad 0 \in U, \quad \text{and} \quad 1 \in U$$
$$\chi_E^{-1}(U) = E, \quad \text{if} \quad 0 \notin U, \quad \text{but} \quad 1 \in U$$
$$\chi_E^{-1}(U) = X \setminus E \quad \text{if} \quad 0 \in U, \quad \text{but} \quad 1 \notin U.$$

In all cases $\chi_E^{-1}(U)$ is measurable. Notice that we did not really use that U is open.

Example: If X and Y are metric spaces and $f : X \longrightarrow Y$ is constinuous, then f is Borel measurable.

In fact, since for U open, $f^{-1}(U)$ is open, it is Borel measurabe.

Theorem 1.1. *Let $(X, \mathcal{M})$ be a measure space, and let Y be a metric (topological)space. Let $f : X \longrightarrow Y$ be a map. Then*
1) If $\Omega = \{E \in Y : f^{-1}(E) \in \mathcal{M}\}$ then Ω is a σ-algebra in Y.
2) If f is measurable and $E \subset Y$ is a Borel set, $f^{-1}(E) \in \mathcal{M}$.
3) If f is measurable, Z is a metric space and $g : Y \longrightarrow Z$ a Borel mapping, and if $h = g \circ f$, then h is measurable.
4) If $Y = [-\infty, \infty]$ then f is measurable if and only if $f^{-1}((\alpha, \infty]) \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$. Similarly, f is measurable if and only if $f^{-1}([-\infty, \alpha)) \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$.

Proof. To prove 1) first observe that $f^{-1}(Y) = X$, so $Y \in \Omega$. Let $A \in \Omega$ and we want to show that $Y \setminus A \in \Omega$. By definition of Ω, $f^{-1}(A) \in \mathcal{M}$. Then $f^{-1}(Y \setminus A) = f^{-1}(Y) \setminus f^{-1}(A) = X \setminus f^{-1}(A) \in \mathcal{M}$. Hence $Y \setminus A \in \Omega$.

Finally notice that

$$f^{-1}(A_1 \cup A_2 \cup A_3 ...) = f^{-1}(A_1) \cup f^{-1}(A_2) \cup f^{-1}(A_3) \cup ...$$

So if $A_j \in \Omega$, $j - 1, 2, ...$, $A_1 \cup A_2 \cup ... \in \Omega$.

Item 2) is a consequence of item 1). To see that just observe that, since f is measurable, Ω contain all open subsets of Y. Since it is a σ-algebra, Ω must contain the Borel sets. Therefore if $E \in \mathcal{B}(Y)$ then $E \in \Omega$ and therefore $f^{-1}(E) \in \mathcal{M}$.

To prove 3) notice that for any subset $V \subset Z$,

$$h^{-1}(V) = f^{-1}(g^{-1}(V)).$$

Let V be open. Since g is Borel measurable, then $g^{-1}(V)$ is a Borel set. By item 2) $f^{-1}(g^{-1}(V)) \in \mathcal{M}$. This proves that h is measurable.

Finally we prove 4). First notice that the open intervals of $Y = [-\infty, \infty]$ are of the form $[-\infty, \alpha)$, (α, β), and $(\alpha, \infty]$. So if f is measurable, then by definition $f^{-1}([-\infty, \alpha) \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$.

Since every open subset of Y is a countable union of disjoint intervals of these types, we just need to show that if $f^{-1}([-\infty, \alpha) \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty])$ and $f^{-1}((\alpha, \beta)) \in \mathcal{M}$ for all $\alpha, beta \in \mathbb{R}$.

Notice that

$$X \setminus f^{-1}([-\infty, \alpha)) = f^{-1}([\alpha, \infty]) \in \mathcal{M}.$$

Since one can write

$$f^{-1}((\alpha, \infty]) = \bigcap_{n=1}^{\infty} f^{-1}([\alpha + 1/n, \infty]) \in \mathcal{M}, \quad \text{and} f^{-1}((\alpha, \beta)) = f^{-1}((\alpha, \infty]) \cap f^{-1}([-\infty, \beta)) \in \mathcal{M}.$$

it follows that $f^{-1}([-\infty, \alpha) \in \mathcal{M}$, $f^{-1}((\alpha, \infty])$, and $f^{-1}((\alpha, \beta)) \in \mathcal{M}$ for all $\alpha, beta \in \mathbb{R}$. Therefore f is measurable.

Item 4) will be very useful throughout this chapter. ☐

1.3. $\limsup$ **and** $\liminf$**.** Before proceeding , we would like to review two important concepts in the convergence of sequences.

Definition 1.4. *Let* $\{a_n \in [-\infty, \infty], \quad n \in \mathbb{N}\}$*. For each* $k \in \mathbb{N}$ *let*

$$b_k = \sup\{a_k, a_{k+1}, ...\}$$
$$c_k = \inf\{a_k, a_{k+1}, ...\}$$

Then

$$b_1 \geq b_2 \geq b_3 \geq ...$$
$$c_1 \leq c_2 \leq c_3 \leq, , , ,$$

We define

$$\limsup_{n\to\infty} a_n = \inf\{b_1, b_2, ...\} = \lim_{k\to\infty} b_k$$
$$\liminf_{n\to\infty} a_n = \sup\{c_1, c_2, ...\} = \lim_{k\to\infty} c_k.$$

Proposition 1.2. *The* $\limsup_{n\to\infty} a_n$ *and* $\liminf_{n\to\infty} a_n$ *satisfy the following:*

$$\limsup_{n\to\infty} a_n = \sup\{\beta : \textit{ there exists a subsequence } a_{n_j} \textit{ of } a_n \textit{ such that } \beta = \lim_{j\to\infty} a_{n_j}\}.$$
$$\liminf_{n\to\infty} a_n = \inf\{\beta : \textit{ there exists a subsequence } a_{n_j} \textit{ of } a_n \textit{ such that } \beta = \lim_{j\to\infty} a_{n_j}\}.$$

Proof. First of all notice that $a_{n_j} \leq b_{n_j}$ and hence $\beta = \lim_{j\to\infty} a_{n_j} \leq \lim_{j\to\infty} b_{n_j} \limsup_{n\to\infty} a_n$. Now we need to show that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ which converges to $\limsup_{n\to\infty} a_n$. Let $A = \overline{\{a_n\}}$ be the closure of the set formed by the elements of the sequence $\{a_n\}$. Then, by definition of closure $b_k \in A$ for $k = 1, 2, ...$ Hence, again by definition of closure, $\limsup_{n\to\infty} a_n \in A$. Therefore there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ which converges to $\limsup_{n\to\infty} a_n$.

The same proof works in the case of $\liminf$. ☐

Theorem 1.2. *A sequence* $\{a_n\}$ *converges if and only if* $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$.

Proof. Suppose $\{a_n\}$ converges, and $L = \lim_{n\to\infty} a_n$. Then the set

$$\{\beta : \text{ there exists a subsequence } a_{n_j} \text{ of } a_n \text{ such that } \beta = \lim_{j\to\infty} a_{n_j}\} = L.$$

Hence $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L$.

Reciprocally, suppose that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L$. Then it is obvious that $\{a_n\}$ converges and its limit is equal to L. ☐

Definition 1.5. *Let* $f_n : X \longrightarrow [-\infty, \infty]$, $n = 1, 2, ...$ *Then we define*

$$(\sup_n f_n)(x) = \sup f_n(x), \quad (\inf_n f_n)(x) = \inf f_n(x),$$
$$\left(\limsup_{n\to\infty} f_n\right)(x) = \limsup_{n\to\infty} f_n(x), \quad \textit{and} \quad \left(\liminf_{n\to\infty} f_n\right)(x) = \liminf_{n\to\infty} f_n(x).$$

The following is an important application of item 4) of Theorem 1.1.

Theorem 1.3. *If* $f_n : X \longrightarrow [-\infty, \infty]$ *is measurable for* $n = 1, 2, ...$ *then*

$$g_s = \sup_n f_n, \quad h_s = \limsup_{n\to\infty} f_n$$
$$g_I = \inf_n f_n, \quad h_I = \liminf_{n\to\infty} f_n$$

are measurable.

Proof. The main step in the proof is to observe that

$$g_s^{-1}((\alpha,\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,\infty]),$$

$$g_I^{-1}([-\infty,\alpha) = \bigcup_{n=1}^{\infty} f_n^{-1}([-\infty,\alpha)).$$

The conclusion of the first part of the theorem follows from item 4) of Theorem 1.1.

Indeed, if $x \in g_s^{-1}((\alpha,\infty])$ then $g_s(x) > \alpha$. Therefore $f_n(x) > \alpha$ for some n and hence, $x \in \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,\infty])$. If $x \in \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,\infty])$, then $f_n(x) > \alpha$ for some n and hence, $g_s(x) > \alpha$.

The proof of the measurability of g_I is identical.

To prove that h_s and h_I are measurable, one just has to use the definition of lim sup and lim inf and that g_s and g_I are measurable.

Just recall that

$$h_s(x) = \inf_{k\geq 1}\left(\sup_{i\geq k} f_i(x)\right) \quad \text{and} \quad h_I(x) = \sup_{k\geq 1}\left(\inf_{i\geq k} f_i(x)\right).$$

We have just shown that $g_{s,k} = \sup_{i\geq k} f_i$ is measurable and that $h_s = \inf_{k\geq 1} g_{s,k}$ is also measurable. $\square$

1.4. **Simple functions.**

Definition 1.6. *A function $f : X \longrightarrow [0,\infty]$ is simple if its range consists of finitely many points $\{\alpha_1, ..., \alpha_N\}$. One can write*

$$f(x) = \sum_{j=1}^{N} \alpha_j \chi_{E_j}, \quad E_j = f^{-1}(\alpha_j).$$

Notice that $X = E_1 \cup E_2 \cup ... \cup E_N$. In view of Example 1.1, f is measurable if and only if E_j is measurable for $j = 1, 2, ..., N$.

Next we show that any function can be approximated by step functions.

Theorem 1.4. *Let $f : X \longrightarrow [0,\infty]$ be measurable. There exist simple measurable functions $s_j : X \longrightarrow [0,\infty)$ such that*

$$1)\ \ 0 \leq s_1 \leq s_2 \leq s_3 \leq ... \leq f$$

$$2)\ \ \lim_{n\to\infty} s_n(x) = f(x), \ \ \forall\, x \in X$$

3) *If f is bounded then the convergence is uniform.*

Proof. For $n = 1, 2, ...$ and $1 \leq i \leq n2^n$, let

$$E_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right), \quad F_n = f^{-1}\left([n,\infty)\right) \quad \text{and}$$

$$s_n = \sum_{j=1}^{n2^n} \frac{i-1}{2^n}\chi_{E_{n,i}} + n\chi_{F_n}.$$

If $f(x) < \infty$, then $f(x) < n$ for some n and moreover $f(x) \in \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$, for some i with $1 \leq i \leq n2^n$. In that case,

$$0 \leq f(x) - s_n(x) \leq 1/2^n. \tag{1.1}$$

If f is bounded, this shows that one can choose n such that (1.1) is satisfied for every x.

If $f(x) = \infty$ then $s_n(x) = n$ and $s_n(x) \to f(x)$ as $n \to \infty$. $\square$

1.5. **measures.**

Definition 1.7. *Let $(X, \mathcal{M})$ be a measurable space. A positive measure μ on $\mathcal{M}$ is a function*

$$\mu : \mathcal{M} \longrightarrow [0, \infty]$$

such that $\mu(\emptyset) = 0$ and if $A_j \in \mathcal{M}$, $j = 1, 2, ...,$ are such that $A_j \cap A_k = \emptyset$, $j \neq k$, then

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j),$$

Theorem 1.5. *Let $(X, \mathcal{M})$ be a measurable space and let μ be a positive measure on $\mathcal{M}$. Then*
1) If $A_j \in \mathcal{M}$, $j = 1, 2, ..., n$ are such that $A_j \cap A_k = \emptyset$, if $j \neq k$, then $\mu(A_1 \cup A_2 \cup ... \cup A_n) = \mu(A_1) + \mu(A_2) + ... + \mu(A_n)$.
2) If $A \subset B$ then $\mu(A) \leq \mu(B)$.
3) If $A_j \in \mathcal{M}$, $j = 1, 2, ...,$ satisfies $A_1 \subset A_2 \subset A_3 \subset ...$ Let $A = \bigcup_{j=1}^{\infty} A_j$, then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.
4) If $A_j \in \mathcal{M}$, $j = 1, 2, ...$ satisfy $A_1 \supset A_2 \supset A_3 \supset ...$ and $\mu(A_1) < \infty$. Let $A = \bigcup_{j=1}^{\infty} A_j$, then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

Proof. Item 1) follows directly from the definition by just taking $A_{n+j} = \emptyset$, $j = 1, 2, ...$ To prove 2) just write $B = A \cup (B \setminus A)$. Then $\mu(B) = \mu(A) + \mu(B \setminus A)$. Since $\mu(B \setminus A) \geq 0$, the result follows.

To prove 3) let $B_1 = A_1$ and $B_j = A_j \setminus A_{j-1}$, $j \geq 2$. Then $B_j \in \mathcal{M}$ and $B_i \cap B_j = \emptyset$ if $i \neq j$.

Now notice that $A_n = \bigcup_{j=1}^{n} B_j$ and therefore $A = \bigcup_{j=1}^{\infty} B_j$. Hence

$$\mu(A_n) = \sum_{j=1}^{n} \mu(B_j), \quad \text{and} \quad \mu(A) = \sum_{j=1}^{\infty} \mu(B_j).$$

Therefore $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

The proof of 4) is an application of 3). Let $C_j = A_1 \setminus A_j$. Then

$$C_1 \subset C_2 \subset C_3.... \quad \text{and } A_1 \setminus A = \bigcup_{j=1}^{\infty} C_j.$$

Since $A_j \subset A_1$, $j = 2, ...,$ $\mu(C_j) = \mu(A_1 \setminus A_j) = \mu(A_1) - \mu(A_j)$. Therefore from 3),

$$\mu(A_1) - \mu(A) = \mu(A_1 \setminus A) = \lim_{j\to\infty} \mu(C_j) = \mu(A_1) - \lim_{j\to\infty} \mu(A_j).$$

This proves 4). □

1.6. **Integration.**

Throughout this section $(X, \mathcal{M})$ is a measurable space and μ is a positive measure on $\mathcal{M}$.

We consider the integration of positive functions, but keeping in mind that any function f can be written as

$$f(x) = f^{+}(s) - f^{-}(-x), \quad f^{+}(x) = \max\{f(x), 0\}, \quad f^{-}(x) = -\min\{f(x), 0\},$$

where $f^{\pm}$ are integrable, and in view of Theorem 1.3 they are also measurable.

First we define the integral of a simple function.

Definition 1.8. *Let* $s : X \longrightarrow [0,\infty]$ *be a non-negative measurable simple function*

$$s(x) = \sum_{j=1}^{N} \alpha_j \chi_{E_j}, \quad \alpha_j \geq 0.$$

Let $E \subset X$ *be a measurable set. Then*

$$\int_E s \; d\mu \overset{\text{def}}{=} \sum_{j=1}^{N} \alpha_j \mu(E \cap E_j).$$

The convention here is that if $\alpha_j = 0$ *and* $\mu(E_j \cap E) = \infty$, $\alpha_j \mu(E \cap E_j) = 0$. *The purpose of this convention is that the integral of the function zero, even if taken over a set of infinite measure, is equal to zero.*

We then define the integral of positive functions:

Definition 1.9. *Let* $f : X \longrightarrow [0,\infty]$ *be a measurable non-negative function and let* $E \subset X$ *be a measurable set. Then*

$$\int_E f \; d\mu \overset{\text{def}}{=} \sup_{s \leq f} \int_E s \; d\mu,$$

where the sup *is taken over all measurable step functions which are less than or equal to* f. *Notice that if* f *is a simple function, the two definitions coincide.*

The following properties of the integral are immediate consequences of the definition:

Proposition 1.3. *The integral satisfies the following properties:*
1) If $f \leq g$, *then* $\int_E f \; d\mu \leq \int_E g \; d\mu$,
2) If $A \subset B$ *and* $f \geq 0$, *then* $\int_A f \; d\mu \leq \int_B f \; d\mu$,
3) $\int_A cf \; d\mu = c \int_A f \; d\mu$, *for any constant* c,
4) If $f)x) = 0$ *for all* $x \in E$, *then* $\int_E f \; d\mu = 0$, *even if* $\mu(E) = \infty$,
5) If $\mu(E) = 0$, *then* $\int_E f \; d\mu = 0$ *even if* $f(x) = 0$ *for all* $x \in E$,
6) If $f \geq 0$, *then* $\int_E f \; d\mu = \int_X f \chi_E \; d\mu$.

The proof is left as an exercise.
First we notice

Theorem 1.6. *Let* $f : X \longrightarrow [0,\infty]$ *be measurable. If* $\int_E f \; d\mu = 0$ *for some* $E \in \mathcal{M}$, *then* $\mu\{x \in E : f(x) > 0\} = 0$.

Theorem 1.7. *Let* $f : X \longrightarrow [0,\infty]$ *be measurable. If* $\int_X f \; d\mu = 0$, *then* $\mu\{x : f(x) > 0\} = 0$.

Proof. Notice that $\{x \in E : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$, $E_n = \{x \in E : f(x) > 1/n\}$, and that $E_1 \subset E_2 \subset \ldots$.

$$\frac{1}{n}\mu(E_n) \leq \int_{E_n} f \; d\mu \leq \int_E f \; d\mu \leq= 0.$$

Thus $\mu(E_n) = 0$. But $\mu(\{x \in E : f(x) > 0\}) = \lim_{n\to\infty} \mu(E_n) = 0$. □

If this definition of integral is good, it must be linear. So we need to prove

Theorem 1.8. *Let* $f : X \longrightarrow [0,\infty]$, *and* $g : X \longrightarrow [0,\infty]$ *be measurable functions. Then*

$$\int_X (f+g) \; d\mu = \int_X f \; d\mu + \int_X g \; d\mu. \tag{1.2}$$

This is not so easy to establish. First we prove it for simple functions:

Proposition 1.4. *Let* s *and* t *be positive measurable simple functions on* X. *Then*

$$\nu(E) = \int_E s\, d\mu \quad \textit{is a measure}$$

$$\int_E (s+t)\, d\mu = \int_E s\, d\mu + \int_E t\, d\mu.$$

Proof. Let

$$s = \sum_{j=1}^{N} \alpha_j \chi_{A_j}, \quad t = \sum_{k=1}^{M} \beta_j \chi_{B_j}.$$

Notice that it is implicit that $X = \bigcup_{j=1}^{N} A_j$ and $X = \bigcup_{j=1}^{M} B_j$ and that $A_i \cap A_j = \emptyset$, $i \neq j$, $B_i \cap B_j = \emptyset$, $i \neq j$.

Let $E = \bigcup_{m=1}^{\infty} E_m$ with $Em \cap E_N = \emptyset$. Then

$$E \cap A_j = \bigcup_{m=1}^{\infty} E_m \cap A_j, \quad \text{and } \mu(E \cap A_j) = \sum_{m=1}^{\infty} \mu(E_m \cap A_j).$$

Therefore

$$\nu(E) = \sum_{j=1}^{N} \alpha_j \mu(A_j \cap E) = \sum_{j=1}^{N} \alpha_j \sum_{m=1}^{\infty} \mu(A_j \cap E_m) = \sum_{m=1}^{\infty} \sum_{j=1}^{N} \alpha_j \mu(A_j \cap E_m) = \sum_{m=1}^{\infty} \nu(E_m).$$

This proves the first claim. To prove the second claim, let

$$E_{ij} = A_i \cap B_j.$$

Then

$$\int_{E_{ij}} (s+t)\, d\mu = (\alpha_i + \beta_j)\mu(E_{ij}).$$

On the other hand it is easy to verify that

$$\int_{E_{ij}} s\, d\mu + \int_{E_{ij}} t\, d\mu = (\alpha_i + \beta_j)\mu(E_{ij}).$$

Since $X = \bigcup_{i,j=1}^{\infty} E_{ij}$ and E_{ij} are pairwise disjoint sets, and $s+t$ is a simple measurable function, it follows from the fisrt claim that

$$\int_E (s+t)\, d\mu = \int_E s\, d\mu + \int_E t\, d\mu.$$

□

To use this result to prove Theorem 1.8 we need a convergence theorem.

1.7. **Convergence Theorems.** As in the previous section, $(X, \mathcal{M})$ is a measurable space and μ is a positive measure on $\mathcal{M}$.

We will prove three convergence theorems. The first and most important one is the monotone convergence theorem. The other two are applications of this theorem.

Theorem 1.9. *(The monotone convergence theorem) Let* f_n *be a sequence of measurable functions on* X, *and suppose that*

$$1)\;\; 0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots \leq \infty \textit{ for every } \; x \in X$$

$$2) \lim_{n\to\infty} f_n(x) = f(x), \;\; \forall \;\; x \in X.$$

Then f is measurable and

$$\lim_{n\to\infty}\int_X f_n\,d\mu=\int_X f\,d\mu.$$

Proof. Since $\int_X f_n\,d\mu \le \int_X f_{n+1}\,d\mu \le \int_X f\,d\mu$, the limit

$$\lim_{n\to\infty}\int_X f_n\,d\mu=\alpha\le\int_X f\,d\mu.$$

Let s be a measurable simple function $0\le s\le f$ and let $c\in(0,1)$. For $n\in\mathbb{N}$, let

$$E_n=\{x: f_n(x)\ge cs(x)\}=(f_n-cs)^{-1}([0,\infty]),\quad n=1,2,...$$

Then E_n is a msasurabel set. Since $f_{n+1}\ge f_n$,

$$E_1\subset E_2\subset ...$$

Finally we have

$$X=\bigcup_{n=1}^{\infty}E_n.$$

To verify this, let $x\in X$. If $f_n(x)-cs(x)<0$, then $f(x)-cs(x)\le 0$, but this is absurd. Hence, $f_n(x)-cs(x)\ge 0$, for some n and thus $x\in E_n$ for some n.

Therefore we have

$$\int_X f_n\,d\mu\ge\int_{E_n} f_n\,d\mu\ge c\int_{E_n} s\,d\mu,\quad n=1,2,....$$

Since by Proposition 1.4

$$\nu(E)=\int_E s\,d\mu$$

is a measure on $\mathcal{M}$, $E_1\subset E_2\subset ...$, and $\bigcup E_n=X$, it follows from item 3) of Theorem 1.5 that

$$\int_X s\,d\mu=\lim_{n\to\infty}\int_{E_n} s\,d\mu.$$

Therefore we conclude that

$$\lim_{n\to\infty}\int_X f_n\,d\mu=\alpha\ge c\int_E s\,d\mu.$$

for every simple measurable function $s\le f$ and for every $c\in(0,1)$. Therefore

$$\alpha\ge\int_X f\,d\mu.$$

This proves the theorem. □

To prove Theorem 1.8 we just need to recall Theorem 1.4. There exist sequences of simple functions s_n and t_n such that

$$0\le s_1\le s_2\le ...\le f,\quad \lim_{n\to\infty}s_n(x)=f(x),$$
$$0\le t_1\le t_2\le ...\le f,\quad \lim_{n\to\infty}t_n(x)=g(x).$$

Then

$$0\le s_1+t_1\le s_2+t_2\le ...\le f+g,\quad \lim_{n\to\infty}s_n(x)+t_n(x)=f(x)+g(x).$$

By Proposition 1.4

$$\int_X (s_n + t_n)\, d\mu = \int_X s_n\, d\mu + \int_X t_n\, d\mu$$

Then (1.2) follows from Theorem 1.9.

The following are consequences of Theorem 1.8 and Theorem 1.9 is

Corollary 1.1. *Let $f_n : X \longrightarrow [0, \infty]$, $n1, 2, ...$ be measurable functions and let*

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad x \in X.$$

Then

$$\int_X f\, d\mu = \sum_{n=1}^{\infty} \int_X f_n\, d\mu.$$

Proof. Just apply Theorem 1.8 and Theorem 1.9 to the sequence $F_N(x) = \sum_{n=1}^{N} f_n(x)$. □

Corollary 1.2. *Let $f : X \longrightarrow [0, \infty]$ be measurable, and let*

$$\nu(E) = \int_E f\, d\mu, \quad E \in \mathcal{M}. \tag{1.3}$$

Then ν is a positive measure on $\mathcal{M}$ and

$$\int_X g\, d\nu = \int_X gf\, d\mu.$$

Proof. Let $E = \bigcup_{j=1}^{\infty} E_j$, $E_j \cap E_k = \emptyset$ if $j \neq k$. One can write

$$\chi_E f = \sum_{j=1}^{\infty} \chi_{E_j} f,$$

where each term $\chi_{E_j} f$ of the series is positive. Then by Corollary 1.1

$$\nu(E) = \int_E f\, d\mu = \int_X \chi_E f\, d\mu = \sum_{j=1}^{\infty} \int_E \chi_{E_j} f\, d\mu = \sum_{j=1}^{\infty} \nu(E_j).$$

This shows that ν is a measure.

To prove the second part, notice that (1.2) holds for $g = \chi_E$, for any $E \in \mathcal{M}$. Hence (1.2) holds for any measurable simple function. For an arbitrary measurable g, let s_n, $n \in \mathbb{N}$ be a monotone sequence of simple functions that converge to g. Since f is positive, $s_n f$ is monotone and converges to fg. Hence the result follows from the monotone convergence theorem. □

Next we prove Fatou's lemma, which really is an application of the monotone convergence theorem

Theorem 1.10. *(Fatou's lemma) Let $f_n : X \longrightarrow [0, \infty]$, $n = 1, 2, ...$ be measurable functions. Then*

$$\int_X (\liminf_{n \to \infty} f_n\, d\mu \leq \liminf_{n \to \infty} \int_X f_n\, d\mu.$$

Proof. Put $g_k(x) = \inf\{f_k, f_{k+1}, ...\}$. Then $g_1 \leq g_2 \leq ...$ and, by definition $\lim_{k \to \infty} g_k(x) = \liminf_{n \to \infty} f_n(x)$. So Theorem 1.9 implies that

$$\int_X \lim_{k \to \infty} g_k\, d\mu = \lim_{k \to \infty} \int_X g_k\, d\mu.$$

We do not know what $\lim_{k\to\infty}\int_X g_k\, d\mu$ is equal to. However, since $g_k \le f_k$, $\int_X g_k\, d\mu \le \int_X f_k\, d\mu$ and we can estimate

$$\lim_{k\to\infty}\int_X g_k\, d\mu \le \liminf_{n\to\infty}\int_X g_n\, d\mu.$$

This proves the theorem □

The last of the three theorems is

Theorem 1.11. *(The dominated convergence theorem) Let* $f_n : X \longrightarrow [0,\infty]$, $n = 1, 2, ...$ *be a sequence of measurable functions. Suppose that* $\lim_{n\to\infty} f_n(x) = f(x)$ *and that there exists a function* $g(x)$ *such that*

$$f_n(x) \le g(x), \quad n = 1, 2, ... \quad \textit{and} \quad \int_X g\, d\mu < \infty.$$

Then

$$\lim_{n\to\infty}\int_X f_n\, d\mu = \int_X f\, d\mu.$$

Proof. Fatou's lemma applied to the sequence f_n gives that

$$\int_X \liminf_{n\to\infty} f_n\, d\mu = \int_X f\, d\mu \le \liminf_{n\to\infty}\int_X f_n\, d\mu.$$

On the other hand, Fatou's lemma applied to the sequence $g_n = g - f_n$ gives

$$\int_X \liminf_{n\to\infty}(g - f_n)\, d\mu \le \liminf_{n\to\infty}\int_X (g - f_n)\, d\mu. \tag{1.4}$$

But $\liminf_{n\to\infty}(g-f_n) = g - \limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty}\int_X (g-f_n)\, d\mu = \int_X g\, d\mu - \limsup_{n\to\infty}\int_X f_n\, d\mu$. Therefore

$$\limsup_{n\to\infty}\int_X f_n\, d\mu \le \int_X \limsup_{n\to\infty} f_n\, d\mu = \int_X f\, d\mu. \tag{1.5}$$

Putting (1.4) and (1.5) together we deduce that

$$\int_X f\, d\mu \le \liminf_{n\to\infty}\int_X f_n\, d\mu \le \limsup_{n\to\infty}\int_X f_n\, d\mu \le \int_X f\, d\mu.$$

This proves the theorem. □

1.8. **Integration of Complex Functions.**

In this section we study the integration of complex valued functions. There is nothing new here, but instead of leaving it all up to the students, I will say something about it.

The first thing to notice is

Proposition 1.5. *A function* $f : X \longrightarrow \mathbb{C}$ *is measurable if and only if* $u = \Re f : X \longrightarrow \mathbb{R}$ *and* $v = \Im f : X \longrightarrow \mathbb{R}$ *are measurable.*

Proof. Let $R = (a,b) \times (c,d)$ be a rectangle with sides parallel to the axes. Then

$$f^{-1}(R) = u^{-1}((a,b)) \cap v^{-1}((c,d)).$$

Let

$$\pi_{\Re} : \mathbb{C} \longrightarrow \mathbb{R}$$
$$\pi_{\Re}(z) = \Re z$$
$$\pi_{\Im} : \mathbb{C} \longrightarrow \mathbb{R}$$
$$\pi_{\Re}(z) = \Im z$$

Be the projections onto the real axis and the imaginary axis respectively. These projections are continuous functions. Notice that

$$u = \pi_{\Re} \circ f, \quad v = \pi_{\Im} \circ f$$

Now let $U \subset \mathbb{R}$ be an open subset. Then

$$u^{-1}(U) = f^{-1}\left(\pi_{\Re}^{-1}(U)\right).$$

Since $\pi_{\Re}^{-1}(U)$ is open, and f is measurable, $u^{-1}(U) \in \mathcal{M}$. $\square$

As mentioned before, one can write $u = u^+ - u^-$, $u^+ = \max\{u, 0\}$, and $u^- = -\min\{f, 0\}$. So if f is complex valued and measurable, we define

$$\int_E f\, d\mu = \left(\int_E u^+\, d\mu - \int_E u^-\, d\mu\right) + i\left(\int_E v^+\, d\mu - \int_E v^-\, d\mu\right).$$

Proposition 1.6. *If $f : X \longrightarrow \mathbb{C}$ is measurable, then $f = u + iv$, then $|f| = \sqrt{u^2 + v^2}$ is measurable.*

Proof. Let $|\cdot| : \mathbb{C} \longrightarrow [0, \infty)$ $|z|$ is the absolute value of z. This is a continuous function. So $|f|^{-1}(U) = f^{-1}\left(|\cdot|^{-1}(U)\right)$ is measurable, if f is measurable and U is open. $\square$

It follows from Theorem 1.6 that if $f, g : X \longrightarrow \mathbb{C}$ are measurable and

$$\int_E |f - g|\, d\mu = 0, \quad E \in \mathcal{M},$$

then $\{x \in E : f(x) \neq g(x)\}$ has measure zero. We say that this property holds almost everywhere.

Let $f : X \longrightarrow \mathbb{C}$ be a measurable function. We define the relation

$$f \sim g \text{ if } \quad \mu(\{x \in E : f(x) \neq g(x)\}) = 0.$$

It is very easy to see that $\sim$ is an equivalence relation and we denote the equivalence class of a function f by $[f]$.

In view of Theorem 1.6 we define

$$L^1(X, d\mu) = \{[f] : X \longrightarrow \mathbb{C} : \int_X |f|\, d\mu < \infty\}$$

Notice that if $f \sim g$ then $\int_X |f - g|\, d\mu = 0$.

We end this section with the following

Theorem 1.12. *Let $\{f_n\}$ be a sequence of complex valued measurable functions defined almost everywhere on X such that*

$$\sum_{n=1}^{\infty} \int_X |f_n|\, d\mu < \infty.$$

Then the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges almost everywhere on X and $f \in L^1(X, d\mu)$, and

$$\int_X f\, d\mu = \sum_{n=1}^{\infty} \int_X f_n\, d\mu.$$

Proof. Let S_n be the domain of f_n. By assumption $\mu(X \setminus S_n) = 0$. Let

$$\phi(x) = \sum_{n=1}^{\infty} |f_n(x)|, \quad x \in S = \bigcap_{n=1}^{\infty} S_n.$$

Notice that $X \setminus S = \bigcup_{n=1}^{\infty}(X \setminus S_n)$. So $\mu(X \setminus S) = 0$. By Corollary 1.1

$$\int_X \phi \, d\mu < \infty. \tag{1.6}$$

Let $E = \{x : \phi(x) < \infty\}$. Then (1.6) implies that $\mu(X \setminus E) = 0$. Hence the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely on E and $|f(x)| \leq \phi(x)$ on E. So $f \in L^1(X, d\mu)$.

Now if $g_N = \sum_{n=1}^{N} f_n$, then $|g_N| \leq \phi$. Since $\mu(X \setminus E) = 0$, the dominated convergence theorem implies that

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

□