Trig Identities

Trig Facts

$\tan x = \dfrac{\sin x}{\cos x}$

$\sec x = \dfrac{1}{\cos x}$

$\operatorname{cossec} x = \dfrac{1}{\sin x}$

(i) $\cos^2 x + \sin^2 x = 1$

(ii) Divide by $\cos^2 x$

$$1 + \tan^2 x = \sec^2 x$$

(iii) $\sin(a+b) = \sin a \cos b + \sin b \cos a.$

$\sin(a-b) = \sin a \cos b - \sin b \cos a.$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$$\cos 2a = \cos^2 a - \sin^2 a.$$

$$\cos 2a = \cos^2 a - (1 - \cos^2 a) = -1 + 2\cos^2 a.$$

$\boxed{\cos^2 a = \dfrac{1 + \cos 2a}{2}}$ $\sin^2 a = \dfrac{1 - \cos 2a}{2}.$

Section 10 Section 7.2

Trigonometric Integrals

$$\frac{d}{dx}\sin x = \cos x \,; \qquad \int \cos x\,dx = \sin x + C$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \int \sin x\,dx = -\cos x + C.$$

$$\tan x = \frac{\sin x}{\cos x}\,; \qquad \frac{d}{dx}\tan x = \frac{\cos^2 x - \sin x(-\sin x)}{(\cos x)^2}$$

$$\boxed{\cos^2 x + \sin^2 x = 1} \qquad = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}.$$

$$\sec x = \frac{1}{\cos x}$$

$$\boxed{\frac{d}{dx} \tan x = \sec^2 x}$$

$$\boxed{\int \sec^2 x \, dx = \tan x + C}$$

$$\int \underline{\tan x} \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u}$$

$$u = \cos x, \quad du = -\sin x \, dx$$

$$\int \frac{du}{u} = \ln|u| + c.$$

$$\int \tan x \, dx = -\ln|\cos x| + c$$

$$= \ln|\cos x|^{-1} + c = \ln\left|\frac{1}{\cos x}\right| + c$$

$$= \ln|\sec x| + c.$$

$$\boxed{\int \tan x \, dx = \ln|\sec x| + c}$$

$$\int \sec x \, dx .$$

$$\frac{d}{dx} \tan x = \sec^2 x .$$

$$\frac{d}{dx} \sec x = \frac{d}{dx} (\cos x)^{-1} = -(\cos x)^{-2} \cdot (-\sin x)$$

$$= \sin x \cdot (\cos x)^{-2}$$

$$= \frac{\sin x}{(\cos x)^2} = \underbrace{\frac{\sin x}{\cos x}}_{\tan x} \cdot \underbrace{\frac{1}{\cos x}}_{\sec x} = \tan x \sec x$$

$$\boxed{\frac{d}{dx} \sec x = \sec x \cdot \tan x}$$

$$\int (\sec^2 x + \sec x \tan x)\, dx = \tan x + \sec x + C$$

$$\int \sec x \, dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx = \int \frac{du}{u}$$

Set $u = \sec x + \tan x$, $du = (\sec^2 x + \sec x \tan x)\, dx$

$$= \ln|u| + C$$

Conclusion:

$$\boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C}$$

Integrals Involving $\sec x$ and $\tan x$.

$$\int \sec^2 x \, dx = \tan x + C.$$

Compute $\int \tan^2 x \, dx$.

We know that

$$\cos^2 x + \sin^2 x = 1.$$

Divide thus by $\cos^2 x$.

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\boxed{1 + \tan^2 x = \sec^2 x}$$

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C.$$

$$\int \tan^3 x \sec^4 x \, dx$$

Even power of $\sec x$.

$$\int \underbrace{\tan^3 x} \cdot \sec^2 x \, dx = \int u^3 \, du = \frac{1}{4} u^4 + C$$

$u = \tan x$, $du = \sec^2 x \, dx$

$$\int \tan^3 x \sec^2 x \, dx = \frac{1}{4} (\tan x)^4 + C.$$

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \cdot \sec^2 x \cdot \underbrace{\sec^2 x \, dx}$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \, (1 + \tan^2 x) \cdot \sec^2 x \, dx$$

$$= \int \tan^3 x \sec^2 x \, dx + \int \tan^5 x \sec^2 x \, dx$$

$$u = \tan x, \quad du = \sec^2 x \, dx$$

$$= \int u^3 \, du + \int u^5 \, du = \frac{1}{4} u^4 + \frac{1}{6} u^6 + c$$

$$\int \tan^3 x \sec^4 x \, dx = \frac{1}{4}(\tan x)^4 + \frac{1}{6}(\tan x)^6 + C$$

Same idea works for odd powers of tangent.

$$\int \tan^3 x \sec^5 x \, dx$$

$$\boxed{\int \sec x \tan x \, dx = \sec x + C}$$

$$\frac{d}{dx} \sec x = \sec x \tan x.$$

$$\int \tan^3 x \sec^5 x \, dx = \int \underbrace{\tan^2 x}_{\sec^2 x - 1} \cdot \sec^4 x \cdot \underbrace{\tan x \sec x \, dx}$$

$$= \int (\sec^2 x - 1) \sec^4 x \cdot \tan x \sec x \, dx$$

$$\sec x = u \quad , \quad du = \sec x \tan x \, dx$$

$$= \int u^4 (u^2 - 1) \, du = \int (u^6 - u^4) \, du$$

$$= \frac{1}{7} (\sec x)^7 - \frac{1}{5} (\sec x)^5 + C.$$

Integrals of products of

$\sin x \quad \cos x$.

$$\int (\cos x)^3 \cdot (\sin x)^5 \, dx \qquad \int (\cos x)^4 \cdot (\sin x)^3 \, dx$$

$$\int \cos x \cdot \sin x \, dx$$

$$\int (\cos x)^2 \sin x \, dx$$

$$\int \underbrace{\sin x.}_{u}\ \underbrace{\cos x\ dx}_{du} = \int u\, du = \frac{1}{2}u^2 + C$$

$$\int \sin x \cos x\, dx = \frac{1}{2}\sin^2 x + C$$

$$\int (\sin x)^3 \cdot (\cos x)^4 dx = \int (\sin x)^2 \cdot (\cos x)^4 \underbrace{\sin x dx}$$

$$\sin^2 x = 1 - \cos^2 x$$

$$= \int (1-\cos^2 x)\cos^4 x \cdot \sin x\, dx$$

$$u = \cos x, \quad du = -\sin x\, dx.$$

$$\boxed{\cos(2a) = \cos^2 a - \sin^2 a}$$

$$\cos^2 a + \sin^2 a = 1 \quad , \quad \cos^2 a = 1 - \sin^2 a$$

$$\cos 2a = 1 - \sin^2 a - \sin^2 a = 1 - 2\sin^2 a$$

$$\boxed{\sin^2 a = \frac{1 - \cos 2a}{2}}$$

$$\cos^2 a = \frac{1 + \cos 2a}{2} .$$

$$-\int (1-u^2)\, u^4 \, du =$$

$$= -\int u^4 \, du + \int u^6 \, du = \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$= \frac{(\cos x)^7}{7} - \frac{(\cos x)^5}{5} + C.$$

$$\int \sin^2 x \, dx = \int \left(\frac{1 - \cos 2x}{2} \right) dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx = \frac{1}{2} x - \frac{1}{4} \sin 2x + C.$$

$$\cancel{\int \tan^5 x\, dx = \int \tan x \left(\sec^2 x - 1\right)^2 dx}$$

$$\cancel{= \int \tan x \left(\sec^4 x - 2\sec^2 x + 1\right) dx}$$

$$= \cdots$$

$$\int \sec^3 x\, dx = \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x\, dx}_{dv}$$

$$u = \sec x\, ; \quad dv = \sec^2 x\, dx$$

$$du = \sec x \tan x\, ; \quad v = \tan x.$$

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x \cdot (\tan x)^2\, dx$$

$$= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx - \int \sec x\, dx$$

$$2\int \sec^3 x = \sec x \tan x - \int \sec x\, dx$$

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x - \ln\left|\sec x + \tan x\right| + C.$$

$$\int \sin 5x \cos 6x \, dx$$

$$\sin(a+b) = \sin a \cos b + \sin b \cos a.$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\sin(a-b) = \sin a \cos b - \sin b \cos a.$$

$$a = 6x; \quad b = 5x$$

$$\sin 11x = \sin 6x \cos 5x + \sin 5x \cos 6x$$

$$\sin x = \sin 6x \cos 5x - \sin 5x \cos 6x$$

$$\sin 6x \cos 5x = \frac{1}{2}(\sin 11x + \sin x)$$

$$\int \sin 6x \cos 5x \, dx = -\frac{1}{22} \cos 11x - \frac{1}{2} \cos x + c.$$