Lesson 11 Section 7.3

Trigonometric Substitutions

Compute $\int_0^1 \sqrt{1-x^2}\,dx$, $\int \frac{x^3}{\sqrt{x^2+9}}\,dx$

$$\int_2^{2\sqrt{2}} \frac{\sqrt{x^2-4}}{x}\,dx.$$

Strategy: Use trigonometric functions To compute these integrals.

Trig Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\boxed{\underbrace{1+\tan^2\theta} = \sec^2\theta.}$$

Idea: x^2+9 $\qquad x = 3\sin\theta$

$$9\sin^2\theta + 9 = 9(\sin^2\theta + 1)$$

But if we set $x = 3\tan\theta$.

$$x^2+9 = 9\tan^2\theta + 9 = 9(\tan^2\theta+1) = 9\sec^2\theta$$

$$\sqrt{x^2+9} = \sqrt{9\sec^2\theta} = 3\sec\theta.$$

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx$$

Substitute

$$x = 3\tan\theta$$

$$dx = 3\sec^2\theta\,d\theta$$

$$\sqrt{x^2+9} = 3\sec\theta$$

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{(3\tan\theta)^3}{\cancel{3}\sec\theta}\,\cancel{3}\sec^2\theta\,d\theta$$

$$= 27\int (\tan\theta)^3 \sec\theta\,d\theta. =$$

$$= 27\int \tan^2\theta \cdot \tan\theta\sec\theta\,d\theta$$

$$\tan^2\theta = \sec^2\theta - 1.$$

$$= 27\int (\sec^2\theta - 1)\underbrace{\tan\theta\sec\theta}\, d\theta$$

$\sec\theta = u$, $du = \sec\theta\tan\theta\, d\theta$

$$= 27\int (u^2 - 1)\, du = 27\left(\frac{u^3}{3} - u\right) + C.$$

$$= 9u^3 - 27u + C.$$

$u = \sec\theta$, $\quad x = 3\tan\theta$, $\quad \tan\theta = x/3$

$\sec^2\theta = 1 + \tan^2\theta \qquad u^2 = 1 + \dfrac{x^2}{9}$

$$u = \sqrt{1 + x^2/9}\,.$$

$$\int \frac{x^3}{\sqrt{x^2+9}}\, dx = 9\left(1+\frac{x^2}{9}\right)^{3/2} - 27\sqrt{1+\frac{x^2}{9}} + C$$

$$\int_0^1 \sqrt{1-x^2}\, dx$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\boxed{\cos^2\theta = 1 - \sin^2\theta}$$

$x = \sin\theta.$ $\quad x=0,\ \theta = 0$; $\quad x = 1,\ \theta = \pi/2$

$$\cos\theta = \sqrt{1-\sin^2\theta}$$

$dx = \cos\theta\, d\theta,$ $\quad \sqrt{1-x^2} = \cos\theta$

$$\int_0^1 \sqrt{1-x^2}\, dx = \int_0^{\pi/2} \cos\theta \cdot \cos\theta\, d\theta = \int_0^{\pi/2} \cos^2\theta\, d\theta.$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

$$\int_0^{\pi/2} \cos^2\theta \, d\theta = \int_0^{\pi/2} \frac{1}{2}\left(1+\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta \, d\theta$$

$$= \pi/4 + \frac{1}{2} \cdot \frac{\sin 2\theta}{2}\Big|_0^{\pi/2}$$

$$= \pi/4.$$

So far: $\sqrt{x^2 + a^2}$ Substitution $x = a\tan\theta$

$\sqrt{a^2 - x^2}$, $x = a\sin\theta$

Next case: $\sqrt{x^2 - a^2}$.

$$\boxed{\begin{aligned} &x = a\tan\theta \\ &x^2 - a^2 = a^2\tan^2\theta - a^2 \\ &= a^2(\tan^2\theta - 1) \end{aligned}}$$

~~[illegible]~~ 1. $\tan^2\theta + 1 = \sec^2\theta$

$$\tan^2\theta = \sec^2\theta - 1.$$

Set $x = a\sec\theta$; $x^2 - a^2 = a^2\sec^2\theta - a^2$

$$= a^2(\sec^2\theta - 1).$$

$$x^2 - a^2 = a^2 \tan^2\theta$$

$$\boxed{\sqrt{x^2 - a^2} = a \tan\theta}$$

$$\int_2^{2\sqrt{2}} \frac{\sqrt{x^2 - 4}}{x}\, dx.$$

$$x = 2\sec\theta$$

$$dx = 2\sec\theta\tan\theta$$

$$\sqrt{x^2 - 4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4(\sec^2\theta - 1)}$$

$$= \sqrt{4\tan^2\theta} = 2\tan\theta.$$

$$\int_2^{2\sqrt{2}} \frac{\sqrt{x^2-4}}{x}\,dx = \int_0^{\pi/4} \frac{2\tan\theta \quad \cancel{2}\,\cancel{\sec}\theta\tan\theta\,d\theta}{\cancel{2}\,\cancel{\sec}\theta}$$

$x = 2\sec\theta$, $\quad x = 2$, $\quad 2 = 2\sec\theta$, $\sec\theta = 1$

$\frac{1}{\cos\theta} = 1$, $\quad \cos\theta = 1$, $\quad \theta = 0$

$x = 2\sqrt{2}$; $\quad 2\sqrt{2} = 2\sec\theta$; $\quad \sqrt{2} = \sec\theta$

$\sqrt{2} = \frac{1}{\cos\theta}$; $\quad \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, $\theta = \pi/4$

$$= \int_0^{\pi/4} 2\tan^2\theta \, d\theta \qquad \tan^2\theta = \sec^2\theta - 1$$

$$= 2\int_0^{\pi/4} (\sec^2\theta - 1)\, d\theta = 2\int_0^{\pi/4} \sec^2\theta \, d\theta -$$

$$2\int_0^{\pi/4} 1 \, d\theta = 2\tan\theta \Big|_0^{\pi/4} - \pi/2$$

$$= 2 - \pi/2 .$$