Lesson 12. Continuation of 7.3

Integrals Involving	Trig Substitution.
$\sqrt{a^2 - x^2}$	$\boxed{x = a\sin\theta}$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$.

$$\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx$$

$x = \frac{3}{5}\sin\theta = 0.6\sin\theta$

$x=0,\ \sin\theta=0,\ \theta=0$

$x=0.6\ \ \sin\theta=1,\ \theta=\pi/2$

<u>Step 1</u>: $\sqrt{9-25x^2} = \sqrt{25\left(9/25 - x^2\right)}$

$= 5\sqrt{9/25 - x^2}.$

$$\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx = \int_0^{0.6} \frac{x^2}{5\sqrt{\frac{9}{25}-x^2}}\,dx.$$

$a^2 = 9/25,\ a = 3/5;$ $\boxed{x = 3/5 \sin\theta.}$

$$dx = \frac{3}{5}\cos\theta \, d\theta$$

$$\int_0^{\pi/2} \frac{\frac{9}{25}\sin^2\theta}{5\sqrt{\frac{9}{25} - \frac{9}{25}\sin^2\theta}} \cdot \frac{3}{5}\cos\theta \, d\theta =$$

$$= \int_0^{\pi/2} \frac{\frac{9}{25}\sin^2\theta}{5 \cdot \cancel{\frac{3}{5}}\sqrt{1-\sin^2\theta}} \cdot \cancel{\frac{3}{5}}\cos\theta \, d\theta =$$

$$= \frac{9}{125}\int_0^{\pi/2} \frac{\sin^2\theta}{\cos\theta} \cdot \cos\theta \, d\theta = \frac{9}{125}\int_0^{\pi/2} \sin^2\theta \, d\theta.$$

$$= \frac{9\pi}{500}.$$

To compute $\int \sin^2\theta \, d\theta$ we use the identity $\sin^2\theta = \frac{1-\cos 2\theta}{2}$.

$$\int_0^{\pi/2} \sin^2\theta \, d\theta = \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2} \right) d\theta$$

$$= \pi/4 - \frac{1}{4}\int_0^{\pi/2} \cos 2\theta \, d\theta = \pi/4 - \frac{1}{4} \sin 2\theta \Big|_0^{\pi/2}$$

$$= \pi/4$$

$$\int_0^1 \sqrt{x - x^2}\, dx$$

$$x^2 - x = (x - 1/2)^2 - 1/4$$

$$x - x^2 = 1/4 - (x - 1/2)^2.$$

$$\int_0^1 \sqrt{x - x^2}\, dx = \int_0^1 \sqrt{1/4 - \underbrace{(x - 1/2)^2}}\, dx$$

set $x - 1/2 = u$, $du = dx$

$$= \int_{-1/2}^{1/2} \sqrt{1/4 - u^2}\, du \;\boxed{= 2 \int_0^{1/2} \sqrt{\tfrac{1}{4} - u^2}\, du}$$

Set $u = \frac{1}{2}\sin\theta$, $du = \frac{1}{2}\cos\theta \, d\theta$

$$\frac{1}{4} - u^2 = \frac{1}{4} - \frac{1}{4}\sin^2\theta = \frac{1}{4}(1 - \sin^2\theta)$$
$$= \frac{1}{4}\cos^2\theta.$$

$$\sqrt{\frac{1}{4} - u^2} = \frac{1}{2}\cos\theta$$

θ when $u = -\frac{1}{2}$; $-\frac{1}{2} = \frac{1}{2}\sin\theta$

$\sin\theta = -1$, $\theta = -\pi/2$.

$u = \frac{1}{2}$; $\frac{1}{2} = \frac{1}{2}\sin\theta$, $\theta = \pi/2$.

$$\int_{-1/2}^{1/2} \underbrace{\sqrt{\frac{1}{4}-u^2}}_{\frac{1}{2}\cos\alpha}\, \underbrace{du}_{\frac{1}{2}\cos\alpha} = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\cos\alpha \cdot \frac{1}{2}\cos\alpha \, d\alpha$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2\alpha \, d\alpha. = \frac{1}{4}\int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2\alpha}{2}\right) d\alpha$$

$$\cos^2\theta = \frac{1+\cos 2\alpha}{2}$$

$$= \frac{1}{8}\int_{-\pi/2}^{\pi/2} (1+\cos 2\alpha)\, d\alpha.$$

$$= \pi/8.$$

$$\int \sqrt{x^2+2x}\, dx = \int \sqrt{(x+1)^2-1}\, dx$$

$$x^2+2x = (x+1)^2-1$$

$$\int \sqrt{(x+1)^2-1}\, dx, \quad \text{Set } \boxed{\begin{array}{l} u = x+1. \\ du = dx \end{array}}$$

$$\int \sqrt{(x+1)^2-1}\, dx = \int \sqrt{u^2-1}\, du.$$

$$u = \sec\theta, \quad u^2-1 = \sec^2\theta - 1$$

$$du = \sec\theta \tan\theta\, d\theta. \quad = \tan^2\theta.$$

$$\int \sqrt{x^2-1}\,dx = \int \tan\theta \cdot \sec\theta \tan\theta \, d\theta$$

$$= \int \sec\theta \cdot \tan^2\theta \, d\theta .$$

$$= \int \sec\theta \left(\sec^2\theta - 1\right) d\theta$$

$$= \int \sec 3\theta \, d\theta - \int \sec\theta \, d\theta .$$

$$\boxed{\int \sec\theta \cdot \tan^2\theta \, d\theta = \int \underbrace{\sec^3\theta \, d\theta} - \int \sec\theta \, d\theta .}$$

$$\underbrace{\int \sec\theta \tan^2\theta \, d\theta} = \sec\theta \tan\theta - \underbrace{\int \sec\theta \tan^2\theta} - \int \sec\theta \, d\theta$$

$$\int \sec^3\theta \, d\theta = \int \underbrace{\sec\theta}_{u} \cdot \underbrace{\sec^2\theta \, d\theta}_{dv} = \underline{\underline{uv}} - \int v \, du$$

$\underbrace{u = \sec\theta}$, $du = \sec\theta \tan\theta \, d\theta$.

$dv = \sec^2\theta \, d\theta$; $\underbrace{v = \tan\theta}$.

$$\int \sec^3\theta \, d\theta = \sec\theta \cdot \tan\theta - \int \sec\theta \tan\theta \cdot \tan\theta \, d\theta$$

$$= \sec\theta \tan\theta - \int \sec\theta \tan^2\theta \, d\theta.$$

$$2\int \sec\theta \tan^2\theta \, d\theta = \sec\theta \tan\theta - \int \sec\theta \, d\theta$$

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + c$$

Conclusión:

$$\int \sec\theta \tan^2\theta \, d\theta = \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + c.$$

$$\sec\theta = x+1. \quad \tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{(x+1)^2 - 1} = \sqrt{x^2+2x}$$

$$\int \sqrt{x^2+2x}\, dx = \frac{1}{2}\cdot(x+1)\cdot\sqrt{x^2+2x} - \frac{1}{2}\ln\left|x+1+\sqrt{x^2+2x}\right| + C.$$