Lesson 14 — Continuation of Section 7.4

Partial Fractions.

degree 1

Example: (1) $$\frac{3x+2}{x^2+4x+3} = \frac{3x+2}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$$

degree 2.

(2) $$\frac{x^2+x+1}{x^2+4x+3}$$

$$\frac{x^2+x+1}{x^2+4x+3} = \frac{(x^2+4x+3) - 3x - 2}{x^2+4x+3}$$

$$= 1 - \frac{3x+2}{x^2+4x+3}$$

$$= 1 + \frac{A}{x+1} + \frac{B}{x+3}.$$

(3) $$\frac{5x+8}{(x+1)(x+2)^2} = \frac{A}{x+1} + \boxed{\frac{B}{x+2}} + \frac{C}{(x+2)^2}$$

$$= \frac{A(x+2)^2 + B(x+2)(x+1) + C(x+1)}{(x+2)^2(x+1)}$$

$$5x+8 = A(x+2)^2 + B(x+2)(x+1) + C(x+1)$$

$$= A(x^2+4x+4) + B(x^2+3x+2) + Cx + C$$

$$5x+8 = (A+B)x^2 + x(4A+3B+C) + 4A+2B+C.$$

$A + B = 0 \qquad B = -A, \qquad B = 5/13$

$2B + C = 3$

$9A + 2C = 1.$

Instead of solving the system. Pick some "convenient" values of x.

<u>$x = -2$</u>: $\quad -5 = (4+9)A; \quad 13A = -5, \quad A = -5/13.$

<u>$x = 0$</u>: $\quad 9A + 2C = 1. \qquad 2C - \frac{45}{13} = 1$

$2C = 1 + \frac{45}{13} = \quad 2C = \frac{58}{13}$

$$\boxed{C = \frac{29}{13}}$$

$A + B = 0$ If $\underline{B=0}$ $A = 0$

$4A + 3B + C = 5$ $C = 5$

$4A + 2B + C = 8$ $C = 8$

$$\frac{3x+1}{(x+2)(x^2+9)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+9}$$

$$= \frac{A(x^2+9) + (Bx+C)(x+2)}{(x+2)(x^2+9)}$$

$$\boxed{3x+1 = A(x^2+9) + Bx(x+2) + C(x+2)}$$

$$= x^2(A+B) + x(2B+C) + 9A + 2C.$$

$$\frac{1}{(x+1)(x^2+4)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

Let us compute some Integrals.

$$\int \frac{x^3+5x+10}{x^2+1}\,dx.$$

$$x^3 = x \cdot x^2 = x(x^2+1-1) = x(x^2+1) - x$$

$$x^3 = x(x^2+1) - x$$

$$x^3+5x+10 = x(x^2+1) - x + 5x + 10 = x(x^2+1) + 4x + 10.$$

$$\frac{x^3+5x+10}{x^2+1} = \frac{x(x^2+1)}{x^2+1} + \frac{4x+10}{x^2+1}$$

$$= x + \frac{4x+10}{x^2+1}.$$

$$\int \frac{x^3+5x+10}{x^2+1}\,dx = \int x\,dx + \int \frac{4x\,dx}{x^2+1} + 10\boxed{\int \frac{dx}{x^2+1}}$$

$$= x^2/2 + 10\tan^{-1}x + 2\ln(x^2+1) + C.$$

$$\int \frac{4x\,dx}{x^2+1} = \int \frac{2\,du}{u} = 2\ln|u| + c$$

$$u = x^2+1\ , \quad du = 2x\,dx$$

$$\int \frac{x^2+8x+1}{(x^2+1)^2}\,dx = \int \frac{x^2+1+8x}{(x^2+1)^2}\,dx$$

$$= \int \frac{x^2+1}{(x^2+1)^2}\,dx + \int \frac{8x}{(x^2+1)^2}\,dx$$

$$= \int \frac{dx}{x^2+1} + \int \frac{8x}{(x^2+1)^2}\,dx$$

$$= \tan^{-1} x - 4(x^2+1)^{-1} + C.$$

$$\int \frac{8x}{(x^2+1)^2}\,dx \underset{u=x^2+1}{=} \int \frac{4\,du}{u^2} = -\frac{4}{u}$$

$$\int \frac{dx}{(x+1)(x^2+4)}.$$

$$\frac{1}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2+4) + (Bx+C)(x+1)}{(x+1)(x^2+4)}$$

$$1 = A(x^2+4) + Bx(x+1) + C(x+1).$$

Set $\underline{x=-1}$: $5A = 1$, $\boxed{A = 1/5}$

Set $x=0$; $4A + C = 1$; $C = 1 - 4A = 1 - 4/5$ $\boxed{C = 1/5}$

Set $\underline{x=1}$:

$$1 = 5A + 2B + 2C \qquad A = \tfrac{1}{5} = C$$

$$1 = 1 + 2B + \tfrac{2}{5} \qquad 2B = -\tfrac{2}{5}$$

$$B = -\tfrac{1}{5}$$

$$\frac{1/5}{x+1} + \frac{-\frac{1}{5}x + \frac{1}{5}}{x^2+1}$$

$$\frac{1}{(x+1)(x^2+4)} = \frac{1}{5(x+1)} + \frac{-x+1}{5(x^2+4)}$$

$$\int \frac{dx}{(x+1)(x^2+4)} = \frac{1}{5}\int \frac{dx}{x+1} - \frac{1}{10}\int \frac{2x}{x^2+4}\,dx + \frac{1}{5}\int \frac{dx}{x^2+4}.$$

$$= \frac{1}{5}\ln|x+1| - \frac{1}{10}\ln(x^2+4) + \frac{1}{10}\tan^{-1}(x/2) + C.$$

$$\int \frac{dx}{x^2+4} = \int \frac{2\sec^2\theta \, d\theta}{4\sec^2\theta} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta$$

$\boxed{x = 2\tan\theta}$, $4 + x^2 = 4 + 4\tan^2\theta$

$= 4\sec^2\theta$

$dx = 2\sec^2\theta \, d\theta$

$\tan\theta = x/2$, $\theta = \arctan(x/2)$.

$$\boxed{\int \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}(x/2) + C.}$$