Lesson 16 Section 7.8

Improper Integrals

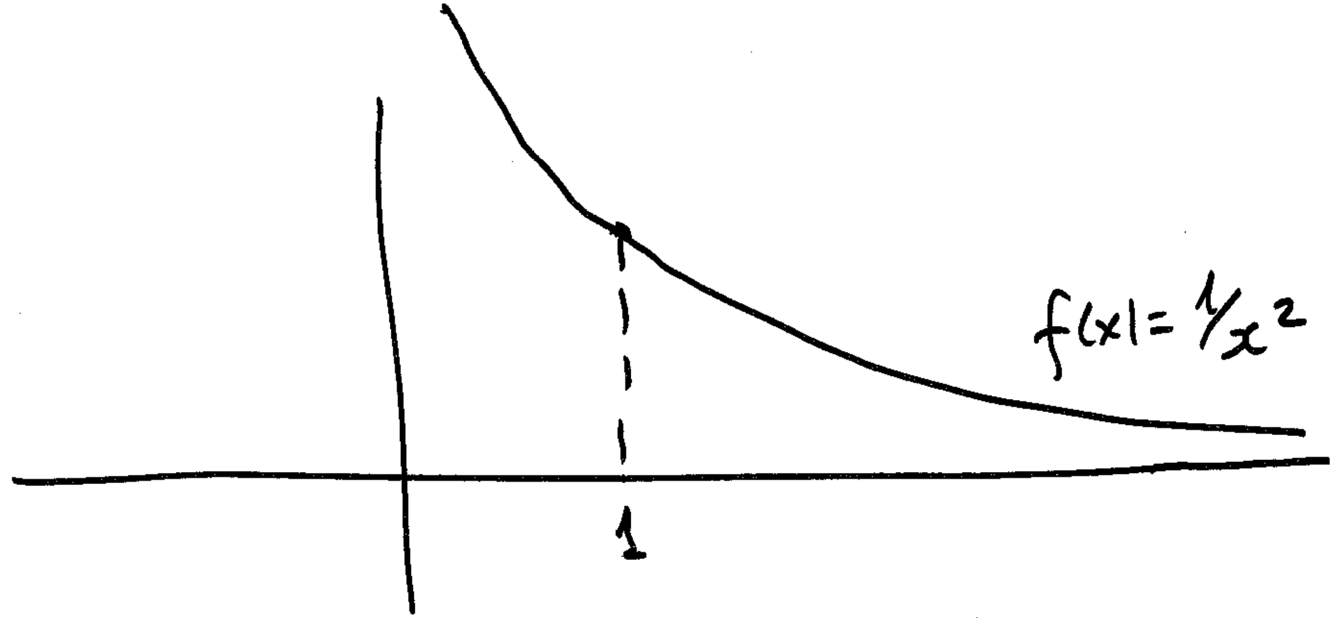
Proper Integrals: $f(x)$ is a continuous function on a finite interval $[a, b]$

$$\int_a^b f(x)\,dx$$

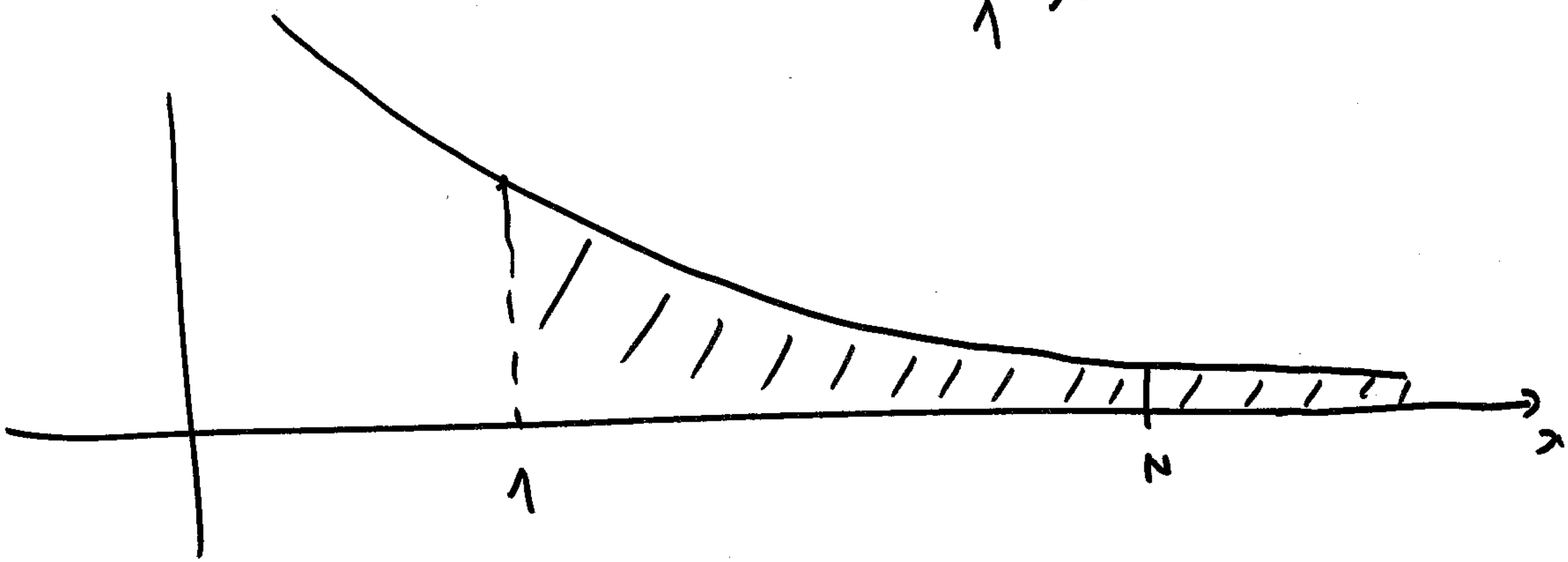
If $f(x)$ is not (piecewise continuous) and bounded or if the interval $[a,b]$ is not finite

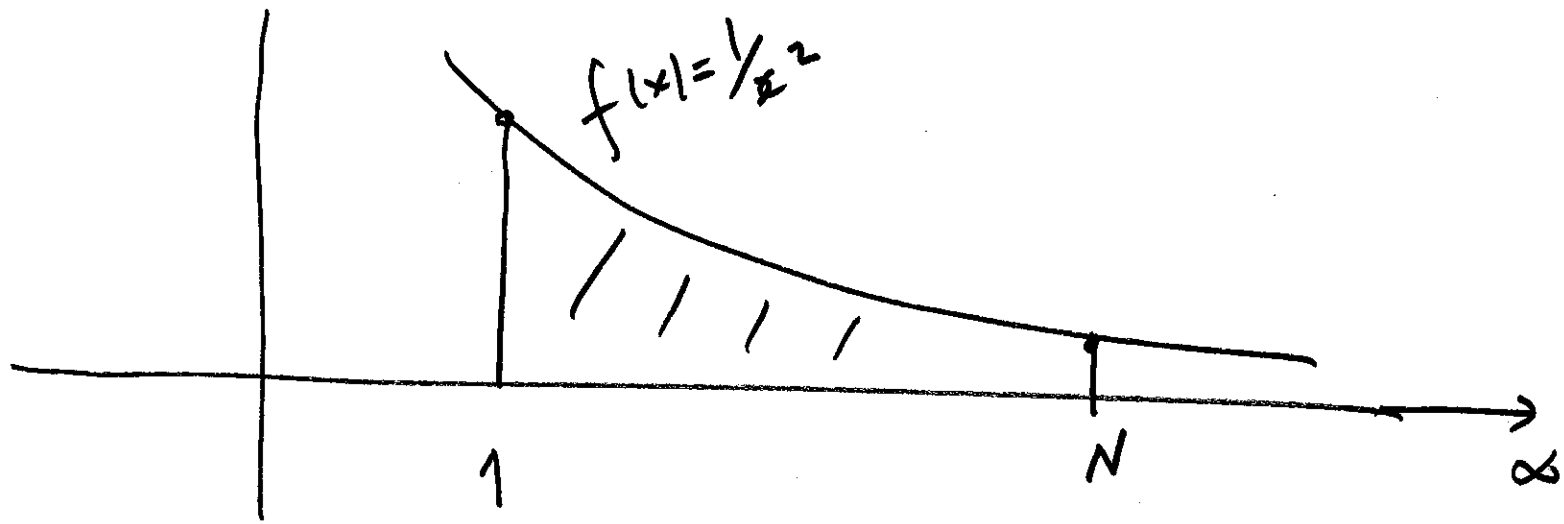
$\int_a^b f(x)dx$ is Improper

Examples: 1)



Can we make sense of $\int_1^\infty \frac{1}{x^2}\,dx$?





$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{x^2}\,dx$$

If this limit exists and is finite, we say that the improper integral converges.

If not, we say it diverges.

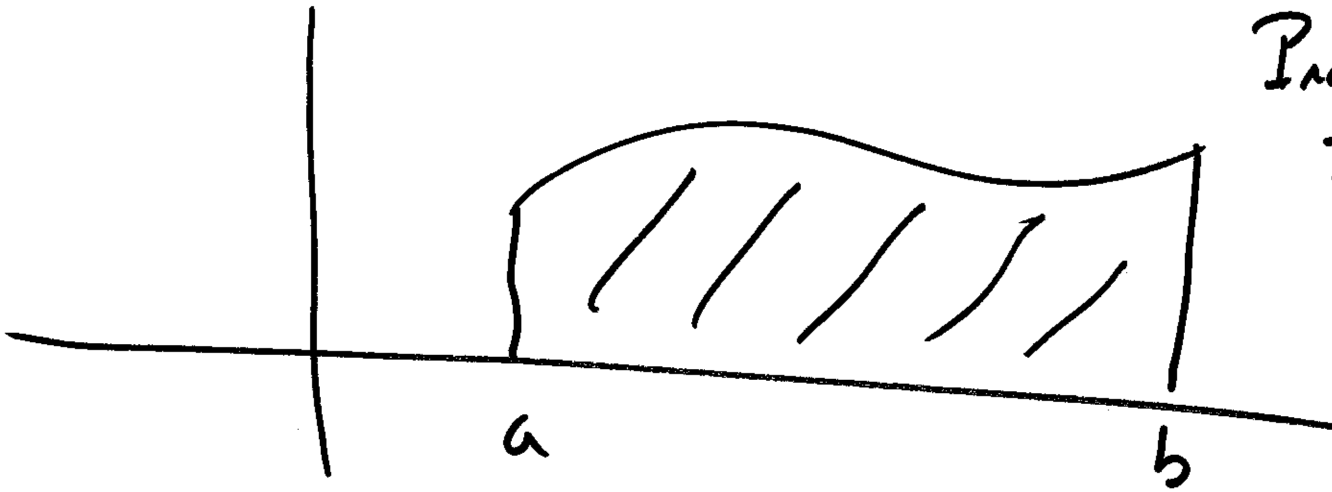
$$\int_1^N \frac{1}{x^2}\,dx = \int_1^N x^{-2}\,dx = -\frac{1}{x}\Big|_1^N$$

$$= 1 - 1/N.$$

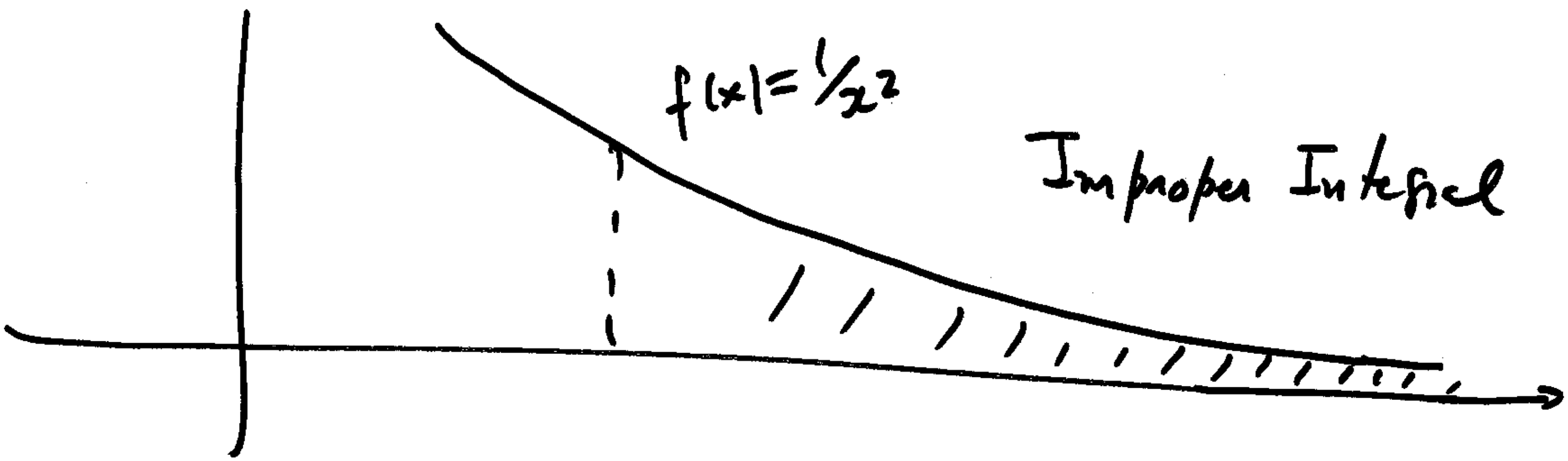
$$\int_1^N \frac{1}{x^2}\,dx = 1 - 1/N.$$

$$\lim_{N\to\infty} 1/N = 0$$

<u>Conclusion:</u> $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{N\to\infty}\int_1^N \frac{1}{x^2}\,dx = 1.$$

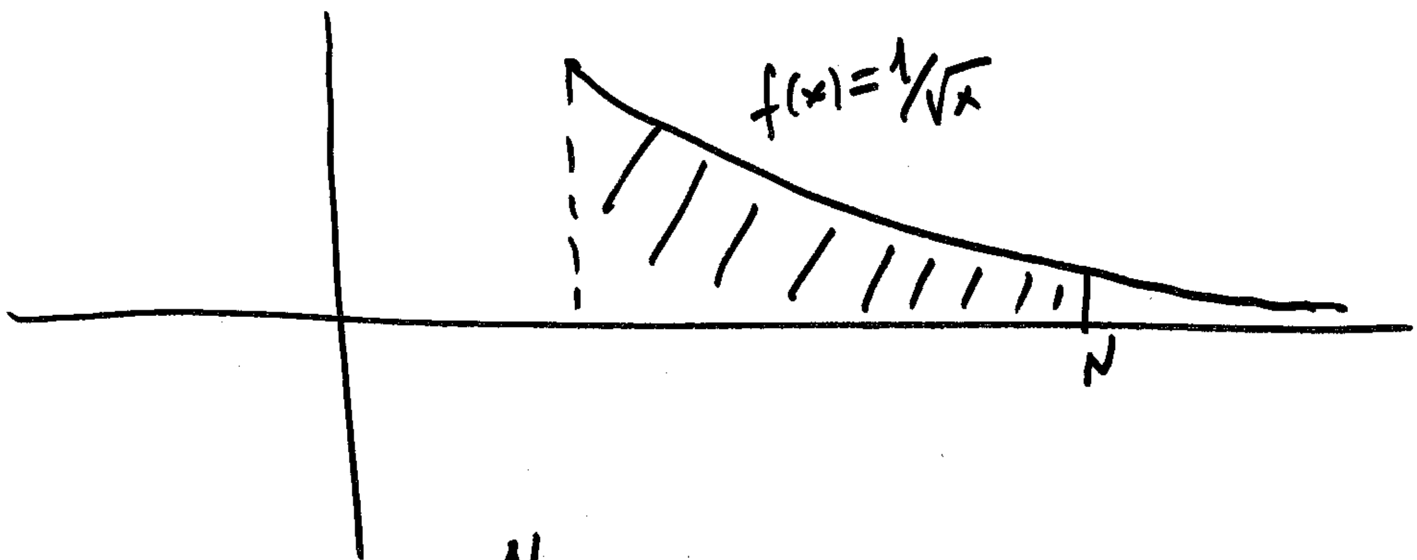


Proper Integral



Improper Integral

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$



$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{\sqrt{x}}\,dx$$

$$\int_1^N \frac{1}{\sqrt{x}}\,dx = \int_1^N x^{-1/2}\,dx = 2x^{1/2}\Big|_1^N = 2\left(\sqrt{N}-1\right).$$

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{\sqrt{x}}\,dx = \lim_{N\to\infty} 2(\sqrt{N}-1) = \infty$$

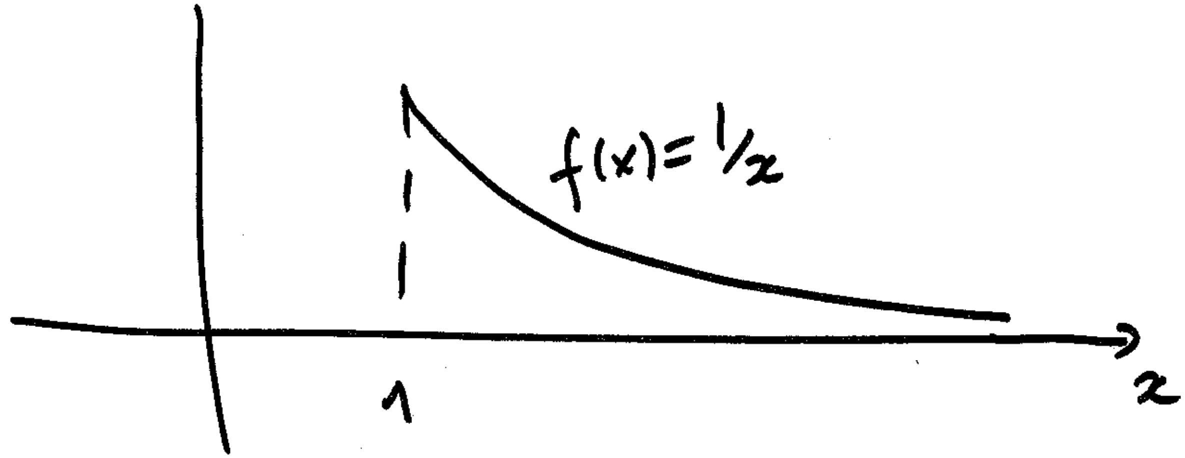
$\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges.

__Question:__ For which values of $p > 0$

does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges?

$p=2$, converges, $p=1/2$, diverges.

$\underline{p=1}$:



$$\int_1^\infty \frac{1}{x}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{x}\,dx = \lim_{N\to\infty} \ln N = \infty$$

$\underline{p\neq 1}$ $$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx =$$

$$= \lim_{N\to\infty} \int_1^N x^{-p}\,dx =$$

$$\int x^{-p}\,dx = \frac{x^{1-p}}{1-p} \quad \text{if } p \neq 1.$$

$$\int_1^N x^{-p}\,dx = \frac{1}{1-p}\, x^{1-p} \Big|_1^N$$

$$= \frac{1}{1-p}\left(N^{1-p} - 1 \right)$$

$$\int_1^\infty x^{-p}\,dx = \lim_{N\to\infty} \frac{1}{1-p}\left(N^{1-p} - 1 \right).$$

If $p < 1$, $1-p > 0$

$$\lim_{N\to\infty} \left(N^{\overbrace{1-p}} - 1\right) = \infty$$

<u>E.g</u>: $p = 1/4$ $\lim_{N\to\infty} \left(N^{3/4} - 1\right) = \infty$

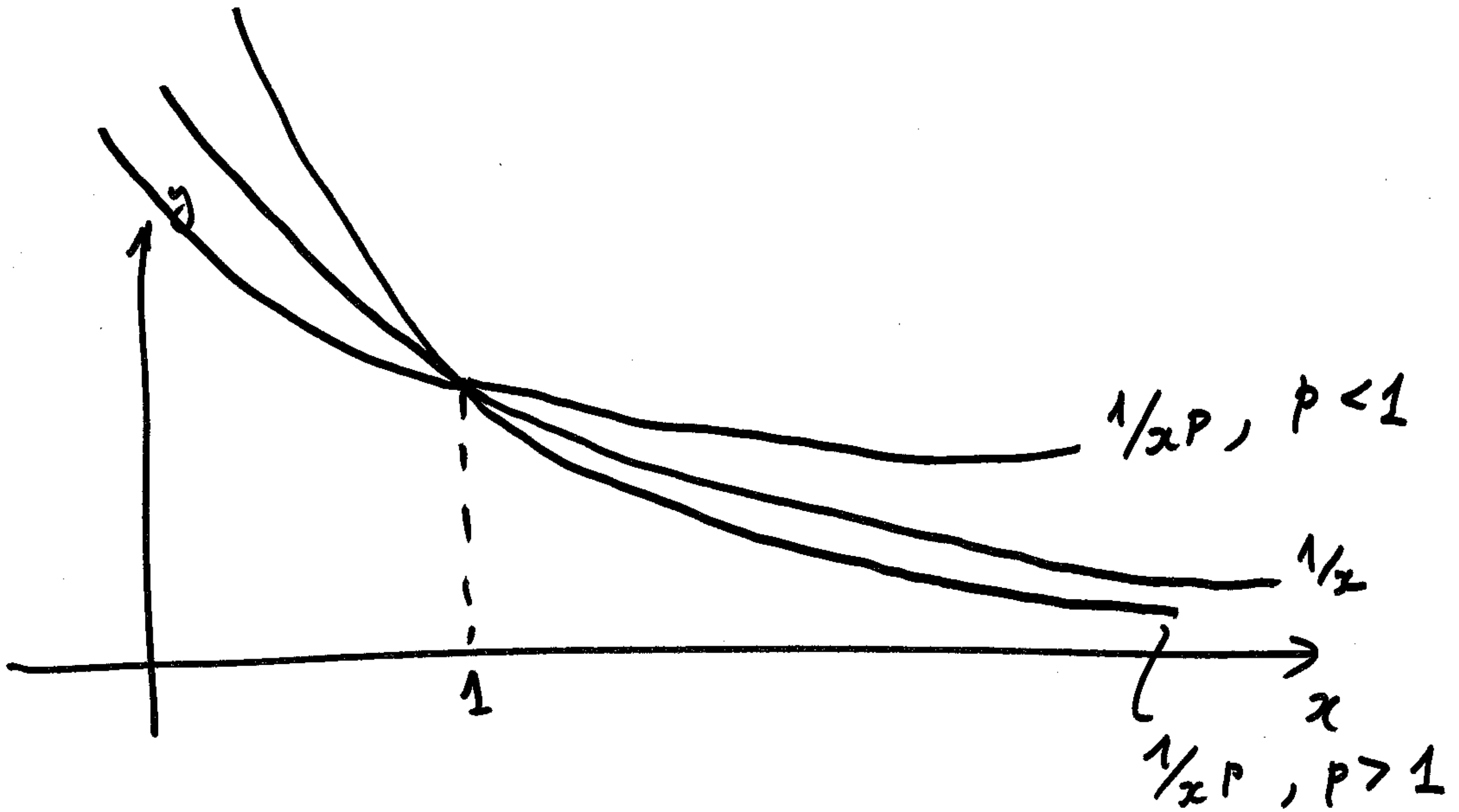
If $p > 1$, $1-p < 0$

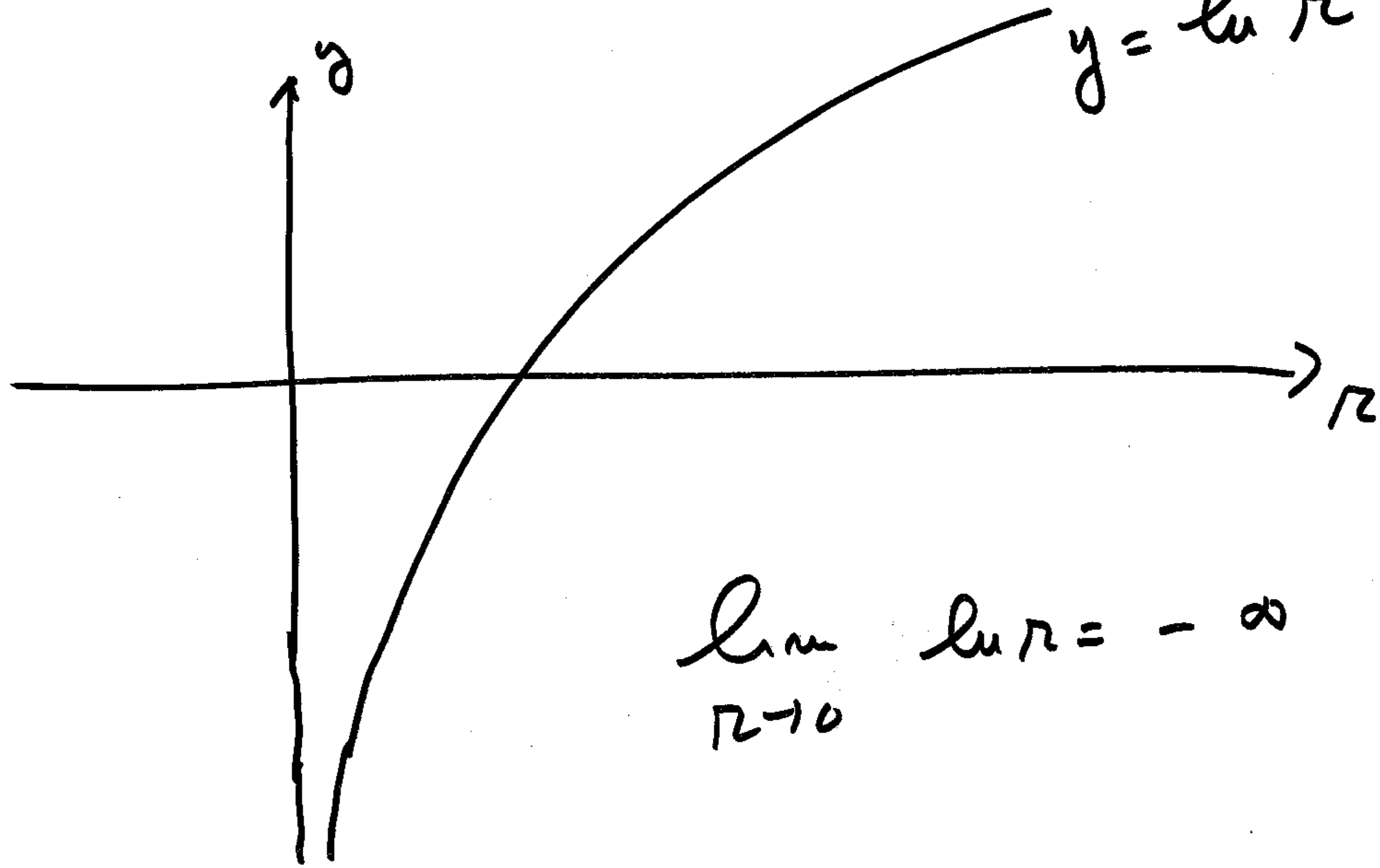
$$\lim_{N\to\infty} \left(N^{1-p} - 1\right) =$$

$p = 2$ $\lim_{N\to\infty} \left(N^{-1} - 1\right) = -1$

Conclusion: $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p>1$

diverges if $p \le 1$.

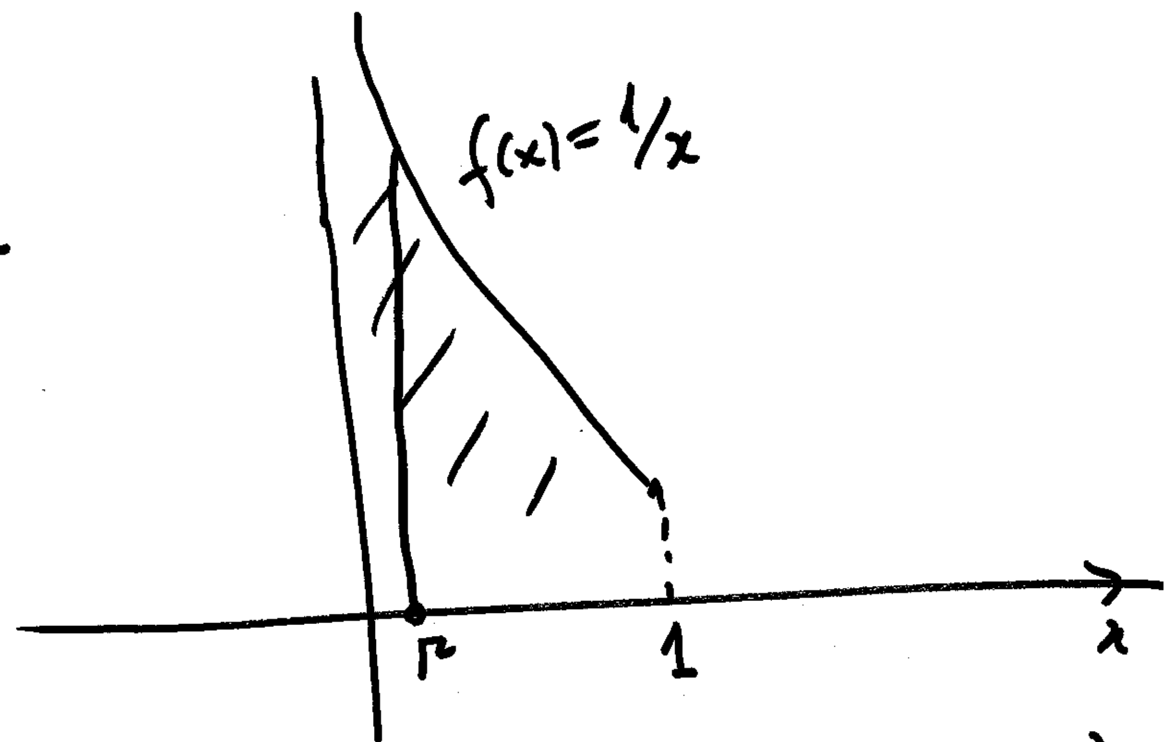




$$\lim_{x \to 0} \ln x = -\infty$$

<u>Case 2</u>: Finite interval, but $f(x)$ blows up.

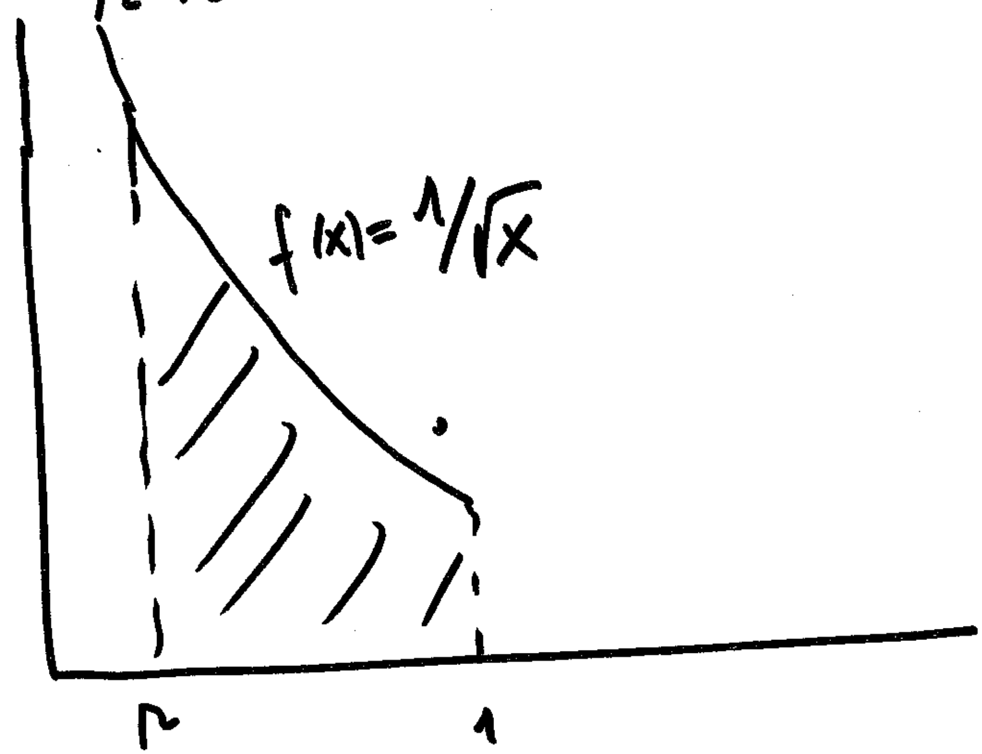
$$\int_0^1 \frac{1}{x}\,dx$$



$$\int_0^1 \frac{1}{x}\,dx = \lim_{r\to 0} \int_r^1 \frac{1}{x}\,dx = \lim_{r\to 0} (\ln 1 - \ln r)$$
$$= \lim_{r\to 0} (-\ln r) = \infty.$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{r\to 0} \int_r^1 x^{-1/2}\,dx$$

$$= \lim_{r\to 0} 2\left(1-\sqrt{r}\right) = 2.$$

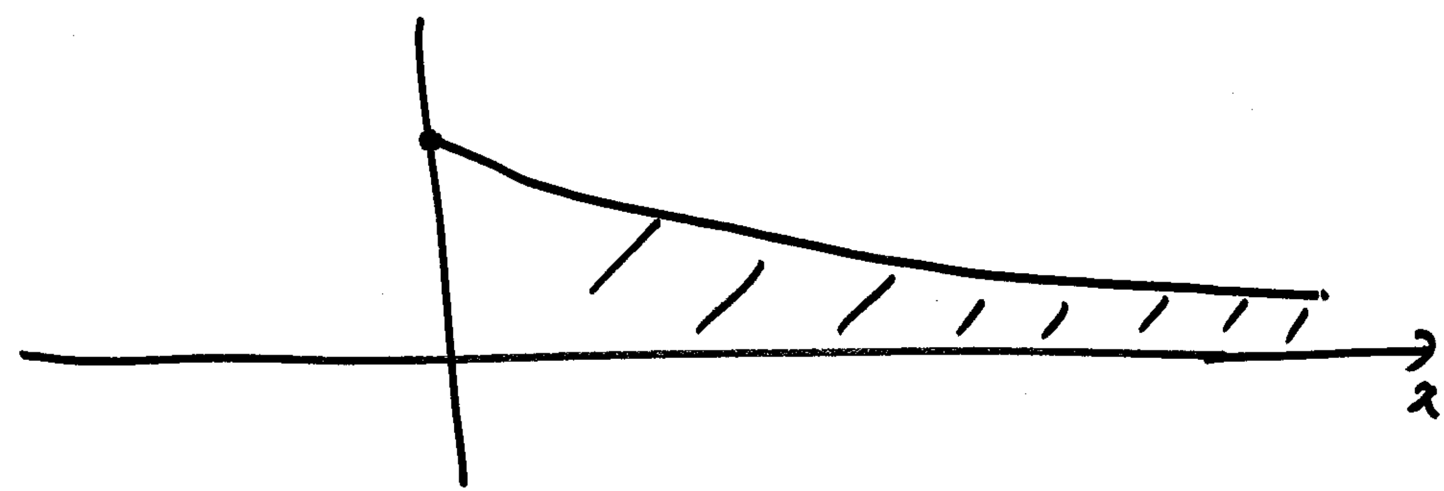


$$\int_0^1 \frac{1}{x^p}\,dx$$ Converges if $p < 1$

Diverges if $p \geq 1$.

<u>Example:</u>

$$\int_0^\infty e^{-x}\,dx = \lim_{N\to\infty}\int_0^N e^{-x}\,dx$$

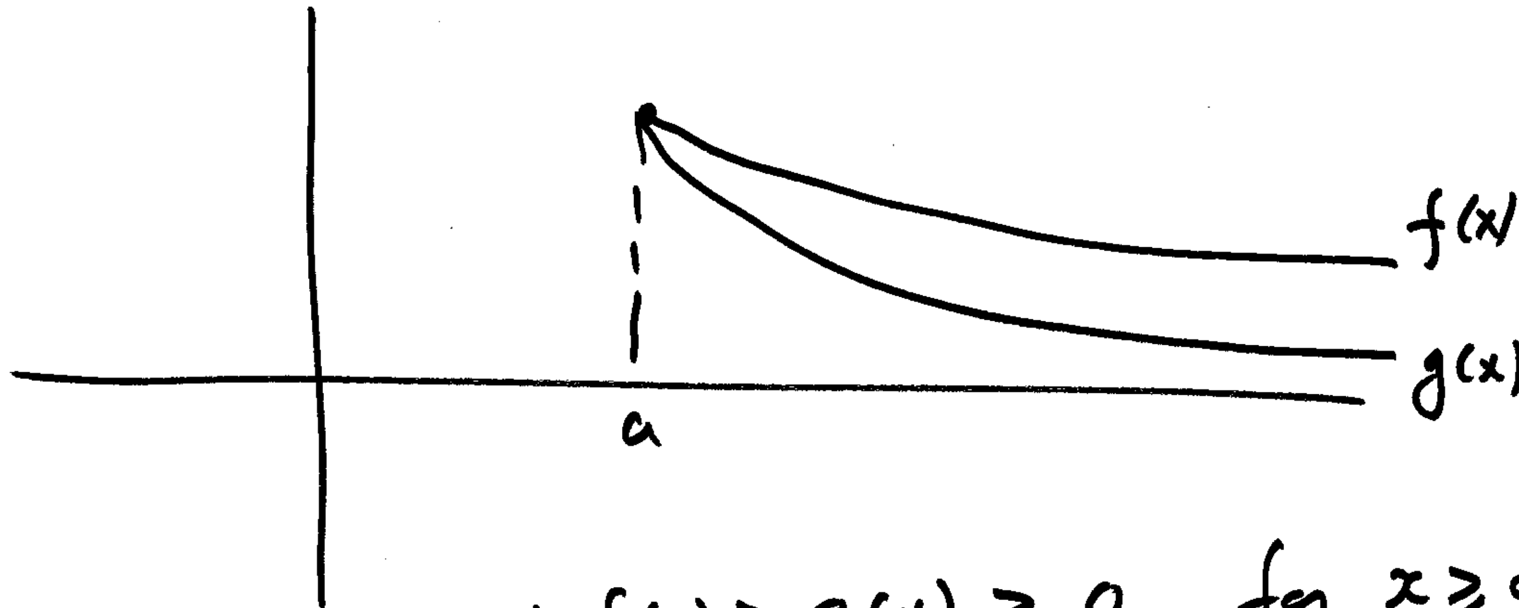


$$\int_0^N e^{-x}\,dx = -e^{-x}\Big|_0^N = -(e^{-N} - e^{-0})$$

$$= e^{-0} - e^{-N} = 1 - e^{-N}$$

$$\int_0^\infty e^{-x}\,dx = \lim_{N\to\infty}(1 - e^{-N}) = 1.$$

<u>Comparison Test for Improper Integrals</u>



$f(x) \geq g(x) \geq 0$ for $x \geq a$.

If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Does $\displaystyle\int_1^\infty \frac{1}{x^5+8x+1}$ converge?

$$x^5+8x+1 > x^5$$

$$\frac{1}{x^5+8x+1} < \frac{1}{x^5}.$$

But I know that $\displaystyle\int_1^\infty \frac{dx}{x^5}$ converges.

By the comparison test $\displaystyle\int \frac{dx}{x^5+8x+1}$ converges.