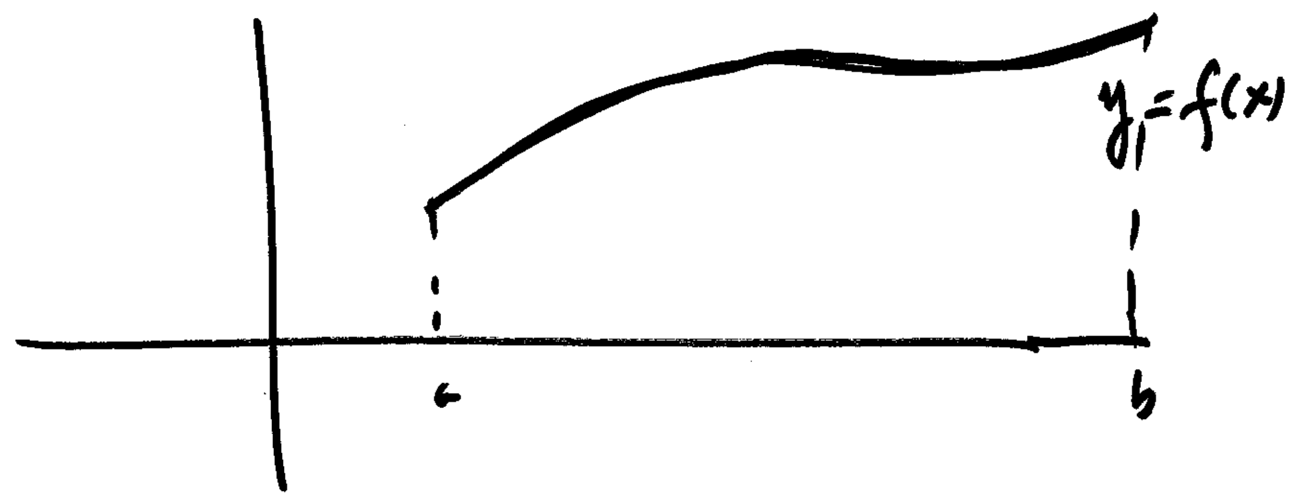
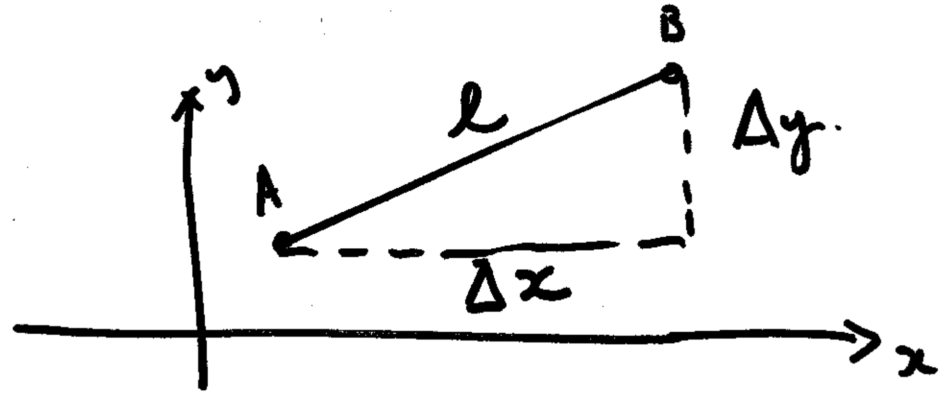
Lesson 17 Sections 8.1 and 8.2

Section 8.1 Arc Length

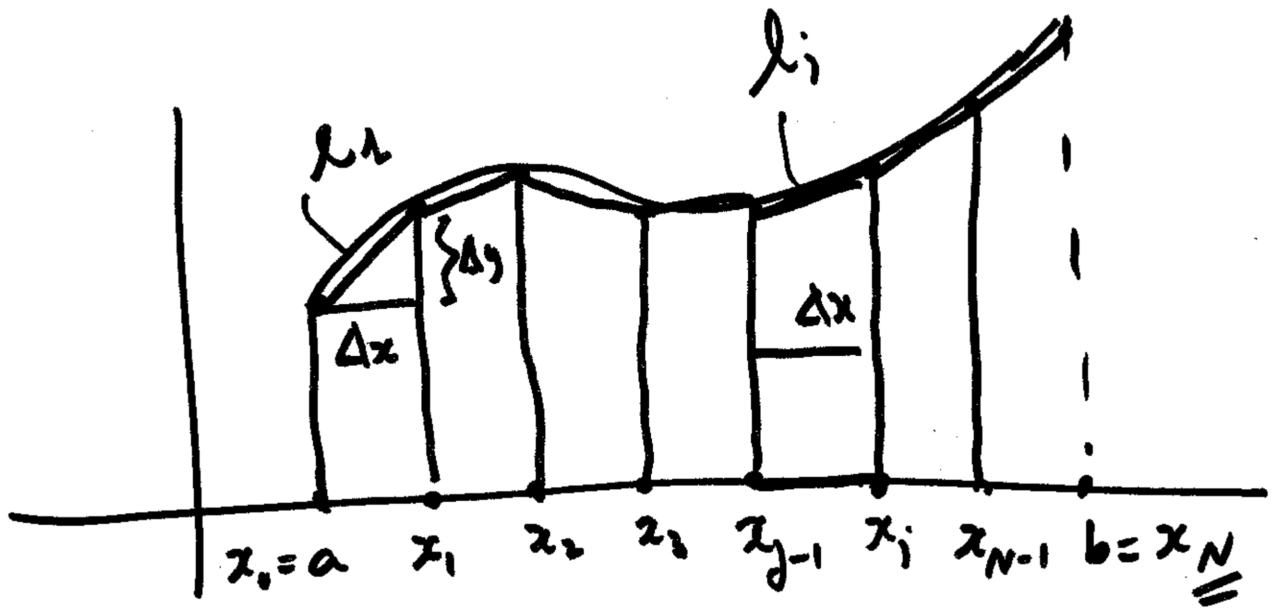


Case 1:



Length of the curve $= l = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

General case

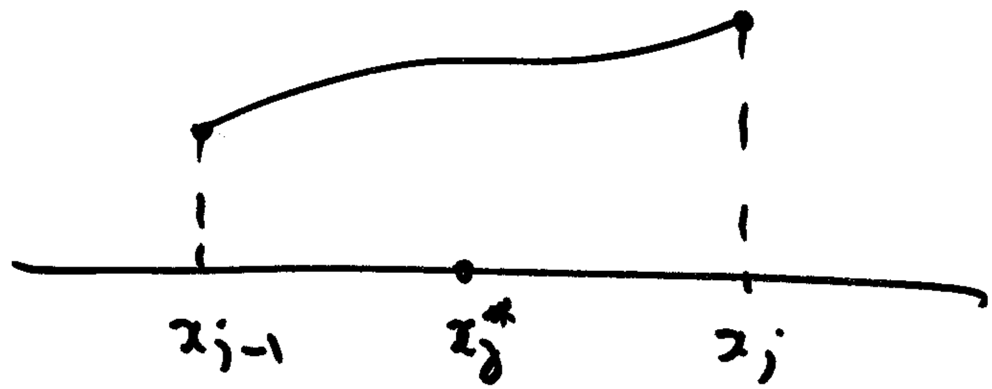


$$l_1 = \sqrt{(\Delta x)^2 + (f(x_1) - f(x_0))^2}$$

$$l_j = \sqrt{(\Delta x)^2 + (f(x_j) - f(x_{j-1}))^2}$$

$$f(x_j) - f(x_{j-1}) = f'(x_j^*) \cdot \underbrace{(x_j - x_{j-1})}$$

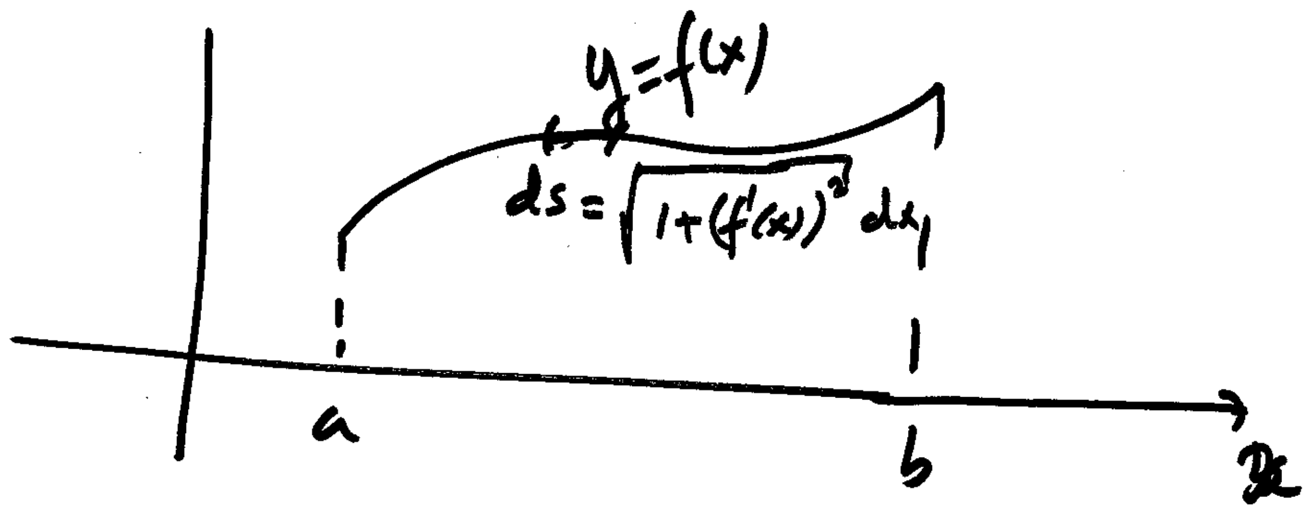
$$= f'(x_j^*) \; \Delta x$$



$$\ell_j = \sqrt{(\Delta x)^2 + (f'(x_j^*))^2 (\Delta x)^2}$$

$$\boxed{\ell_j = \sqrt{1 + (f'(x_j^*))^2} \; \Delta x.}$$

Conclusion:

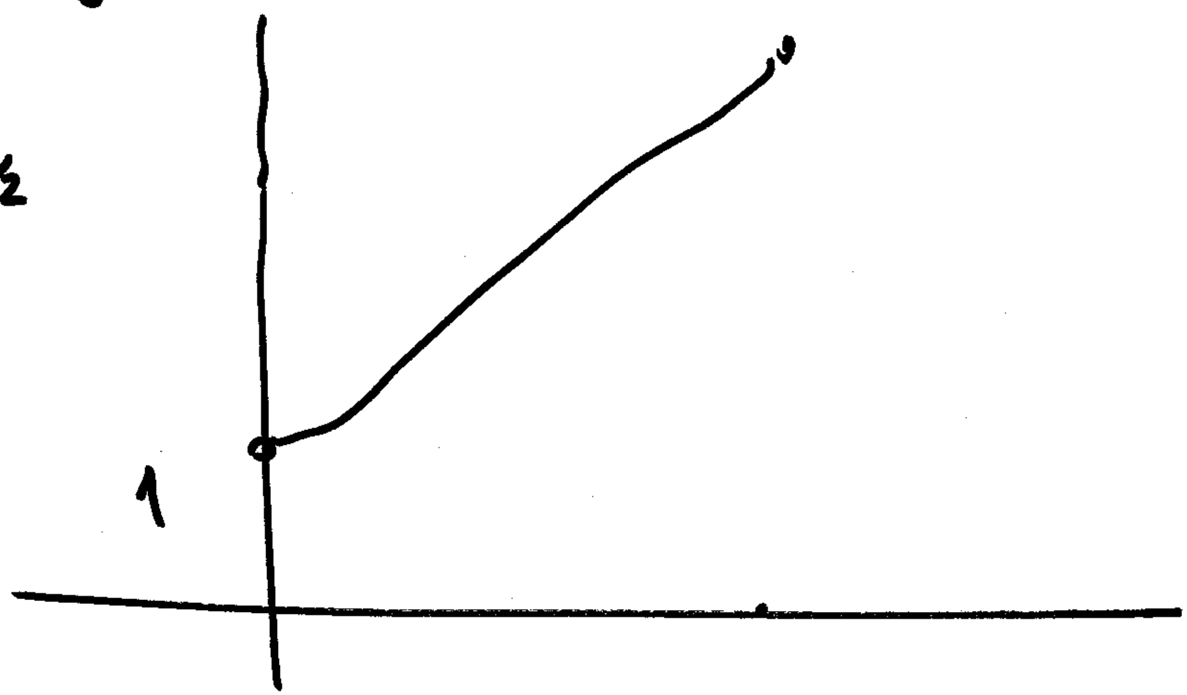


The length of the curve =

$$\int_a^b \sqrt{1+(f'(x))^2}\;dx.$$

Example: Find the length of the curve

$$y = 1 + 6x^{3/2} \qquad 0 \le x \le 1.$$



$f(x) = 1 + 6x^{3/2}$

$f'(x) = 6 \cdot \frac{3}{2} x^{1/2}$

$= 9x^{1/2}$

$$L = \int_0^1 \sqrt{1 + (9x^{1/2})^2}\, dx$$

$$L = \int_0^1 \sqrt{1+81x}\ dx \qquad u = 1+81x$$

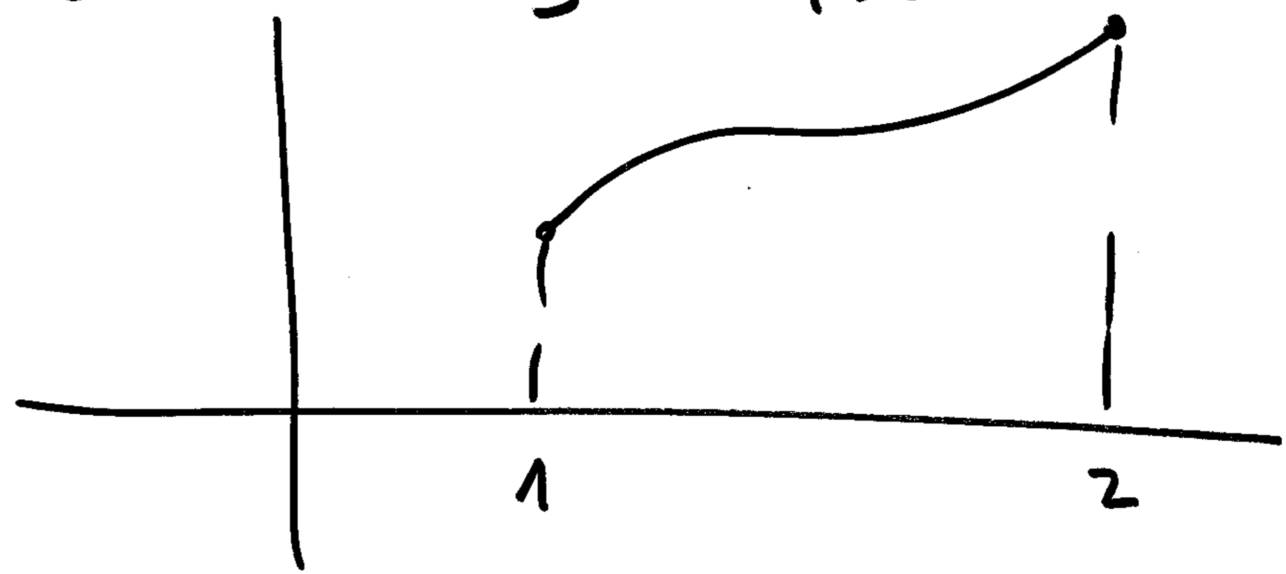
$$du = 81\,dx$$

$$L = \int_1^{82} u^{1/2}\ \frac{du}{81} = \frac{1}{81}\cdot\frac{2}{3}\,u^{3/2}\Big|_1^{82}$$

$$= \frac{2}{243}\left((82)^{3/2} - 1\right).$$

$y = x^3/3 + 1/4x \qquad 1 \le x \le 2.$

$f(x) = x^3/3 + 1/4x = \dfrac{x}{3}^3 + \dfrac{1}{4x}$



$f'(x) = x^2 - 1/4x^2.$

$$L = \int_1^2 \sqrt{1 + \left(x^2 - 1/4x^2\right)^2}\, dx.$$

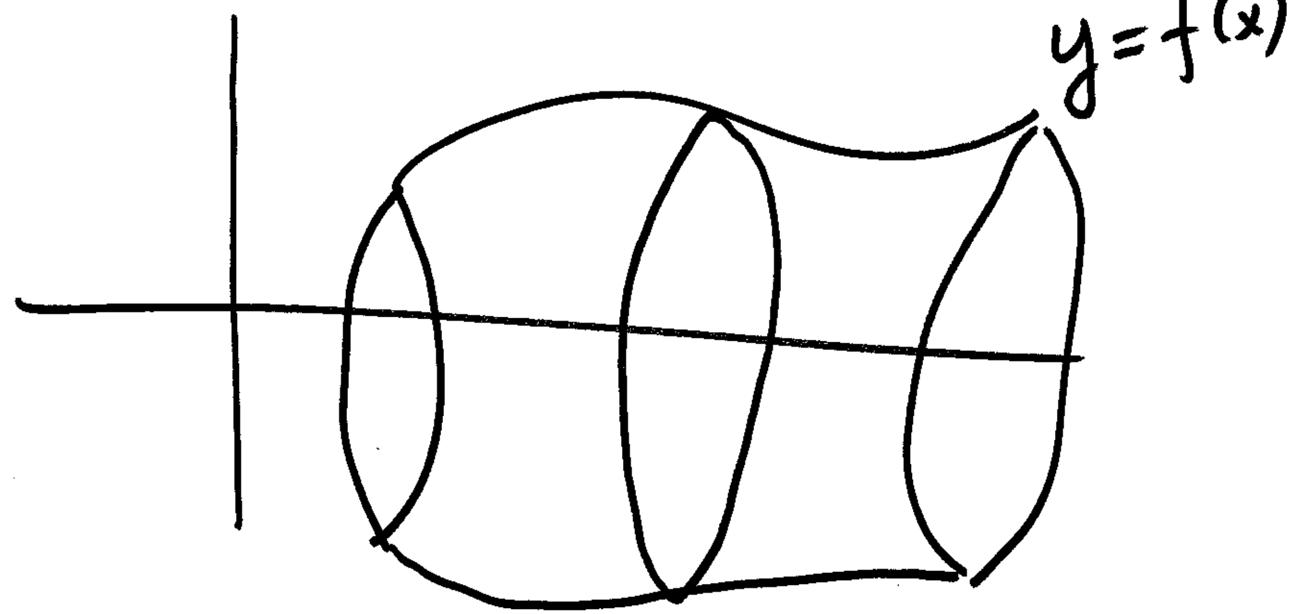
$$1 + \left(x^2 - \frac{1}{4x^2}\right)^2 = 1 + x^4 - \frac{2}{4} + \frac{1}{16x^4}$$

$$= x^4 + \frac{1}{2} + \frac{1}{16x^4}$$

$$= \left(x^2 + \frac{1}{4x^2}\right)^2.$$

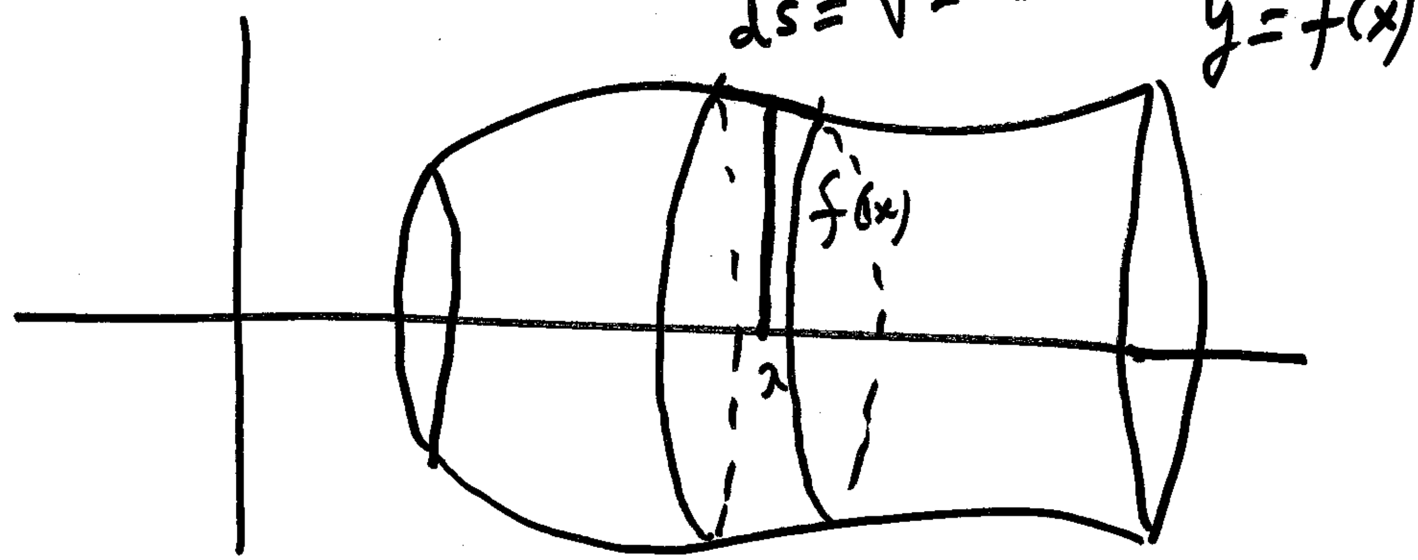
$$\sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2} = x^2 + \frac{1}{4x^2}.$$

$$L = \int_1^2 \left(x^2 + \frac{1}{4x^2}\right) dx = \left(\frac{x^3}{3} - \frac{1}{4x}\right)\Big|_1^2 = \ldots$$

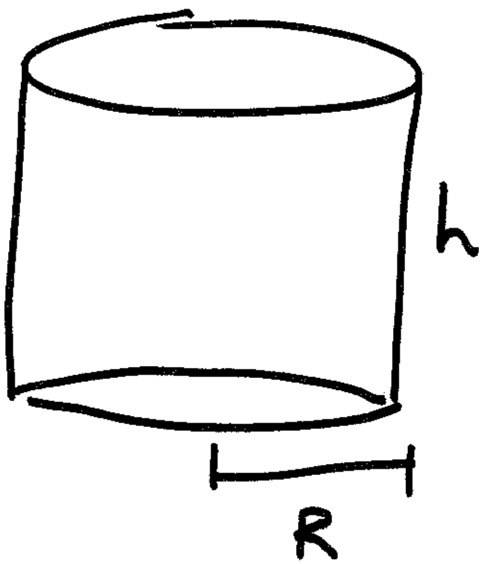


Find the area of the surface obtained by rotating the curve $y=f(x)$ about the x-axis.

$$ds = \sqrt{1+(f'(x))^2}\,dx$$



$$dA = 2\pi \underbrace{f(x)}_{\text{Radius}} \underbrace{\sqrt{1+(f'(x))^2}}_{\text{height}}\,dx$$

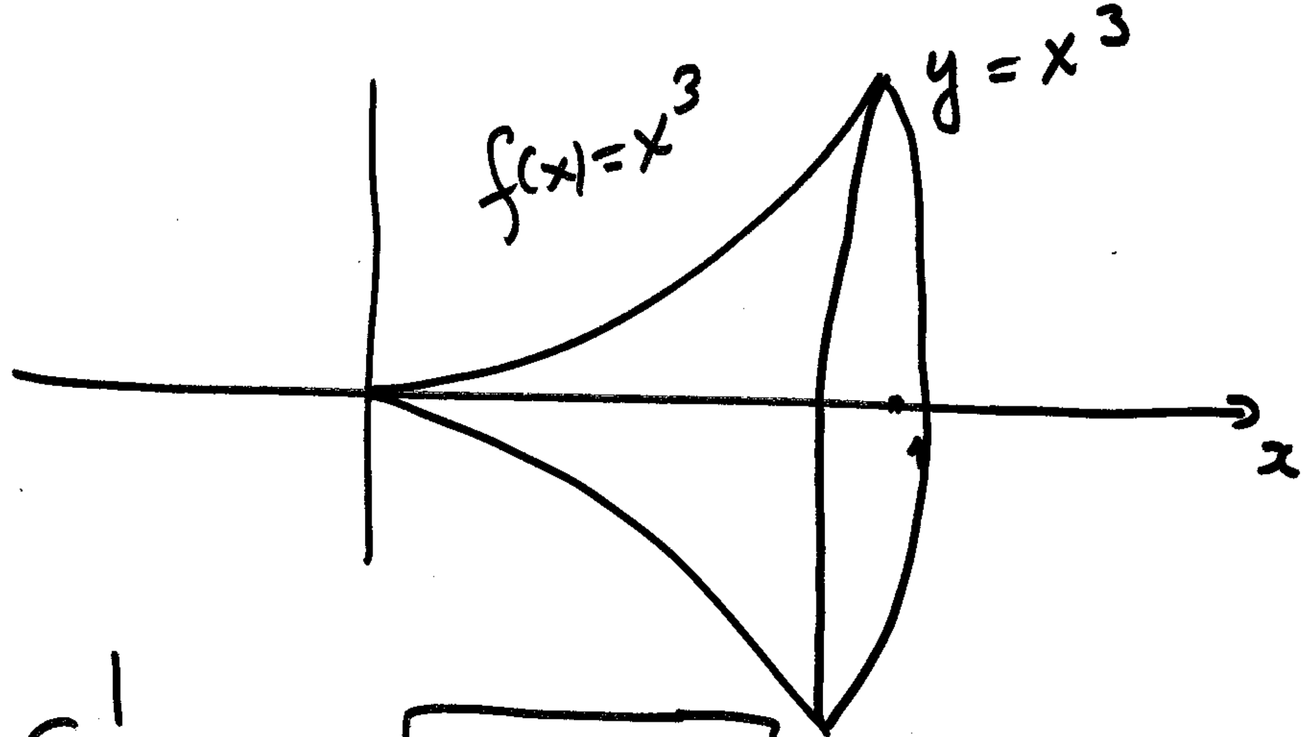


$$\text{Area} = 2\pi R \underset{\sim}{h}.$$

Area of the Surface:

$$\boxed{A = \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx}$$

Example:



$$A = 2\pi \int_0^1 f(x) \sqrt{1+(f'(x))^2}\, dx$$

$$f'(x) = 3x^2.$$

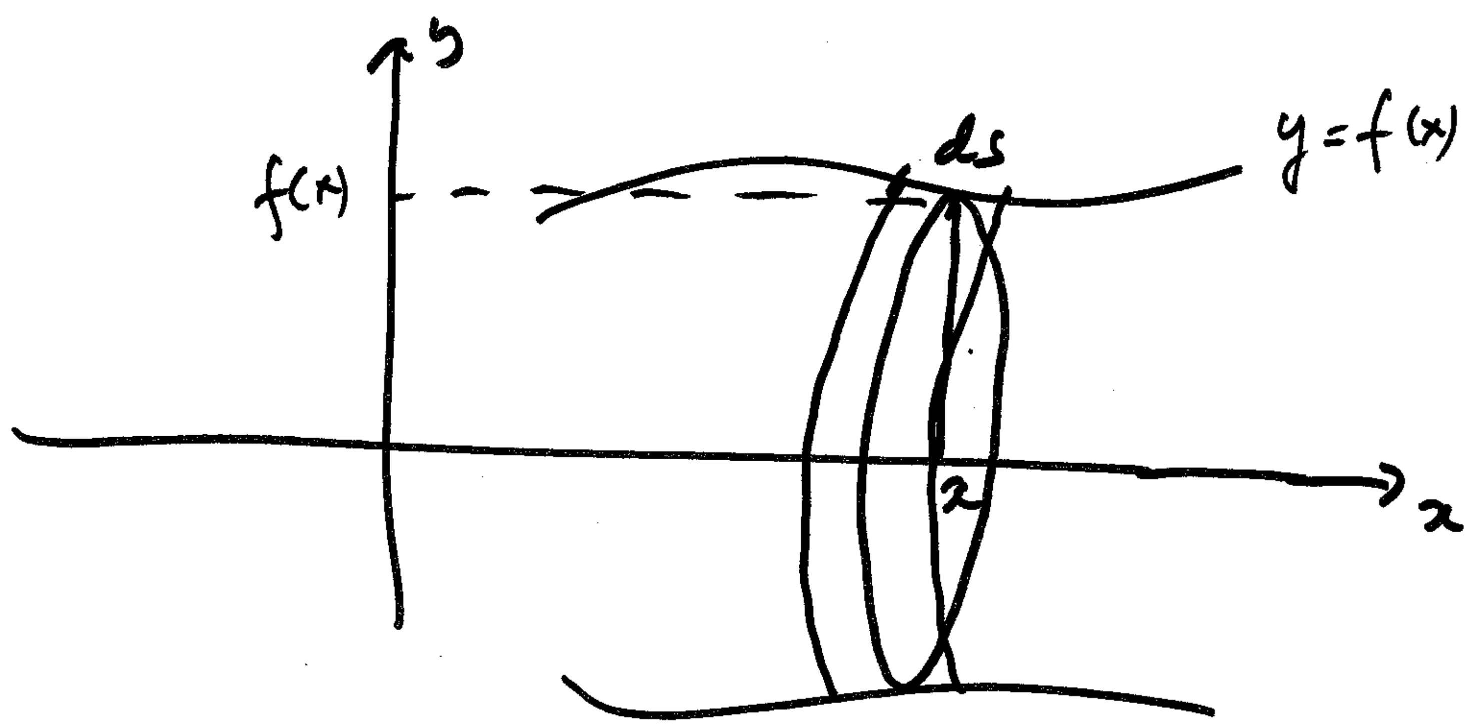
$$A = 2\pi \int_0^1 x^3 \underbrace{\sqrt{1+9x^4}}_{\sqrt{u}}\, dx$$

$$u = 1 + 9x^4, \quad du = 36x^3\,dx$$

$$A = 2\pi \int_1^{10} u^{1/2} \frac{du}{36}$$

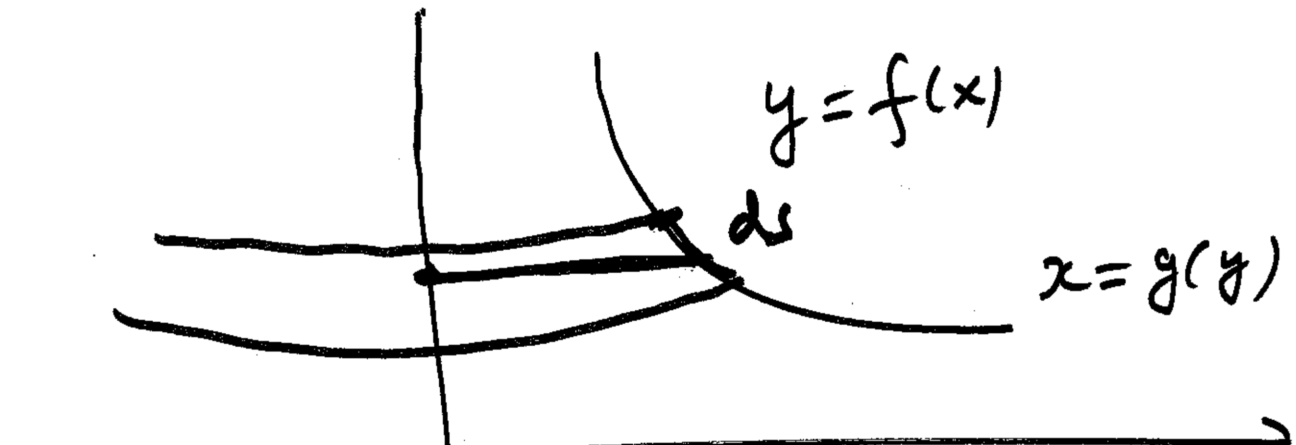
$$= \frac{\pi}{18} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10}$$

$$= \frac{\pi}{27} \left((10)^{3/2} - 1 \right).$$



$$dA = 2\pi \cdot y \, ds$$

$$\boxed{A = 2\pi \int \underbrace{y}\, \underbrace{ds}.}$$

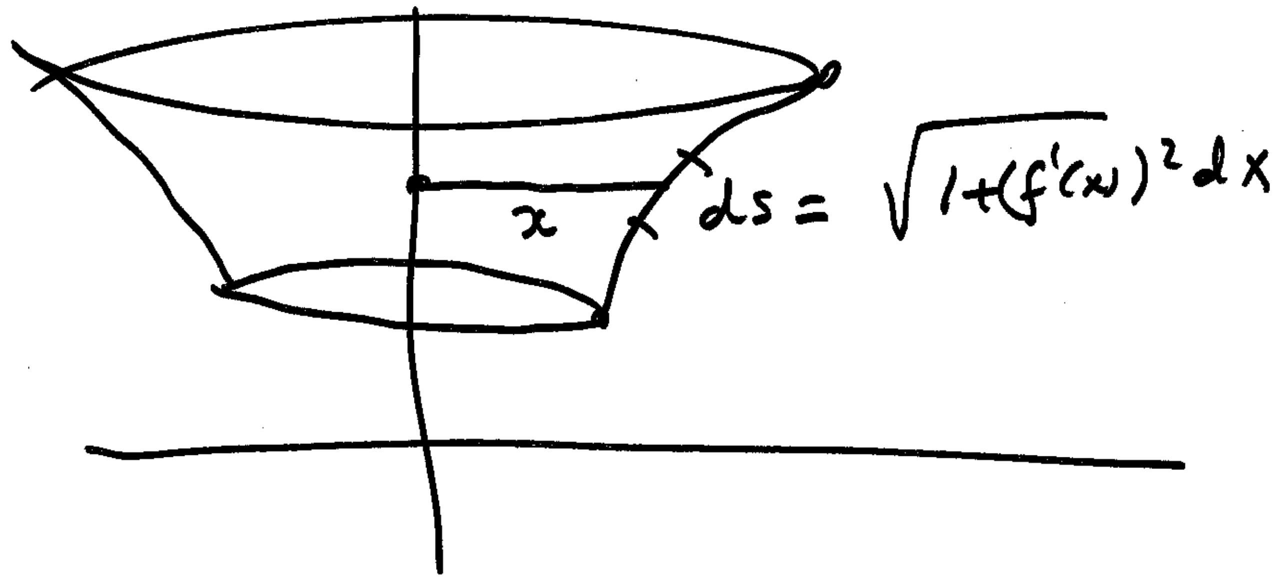


$$ds = \sqrt{1+(f'(x))^2}\,dx$$

$$\text{Area} = 2\pi \int x\,ds$$

$$= 2\pi \int x \sqrt{1+(f'(x))^2}\,dx$$

Example: $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x \qquad 1 \le x \le 2$



$\cancel{y = g(x}$ $\quad x = g(y)$

$$A = 2\pi \int g(y)\sqrt{1+(g'(y))^2}\,dy.$$

Instead of doing this, we

$$A = 2\pi \int_1^2 x \sqrt{1 + (f'(x))^2}\, dx$$

$$2\pi \int_1^2 x \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\, dx$$

$$= 2\pi \int_1^2 x \left(\frac{x}{2} + \frac{1}{2x}\right) dx = \ldots$$