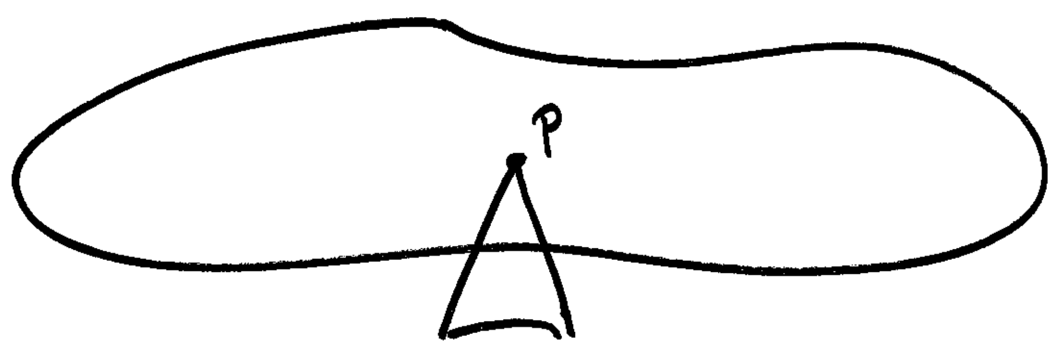
Lesson 18 Section 8.3

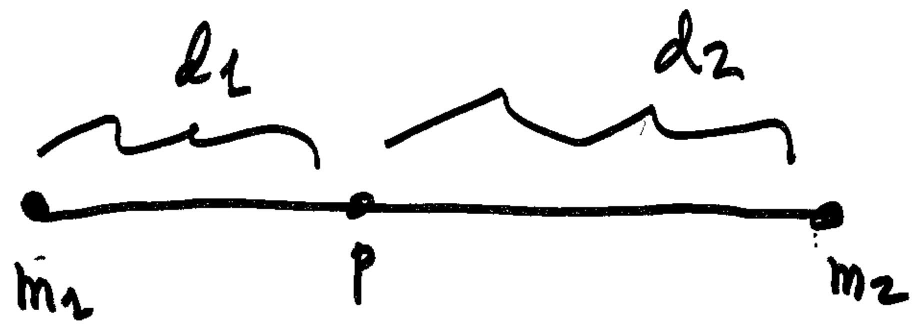
Center of Mass (Centroid)



Thin plate (lamina)

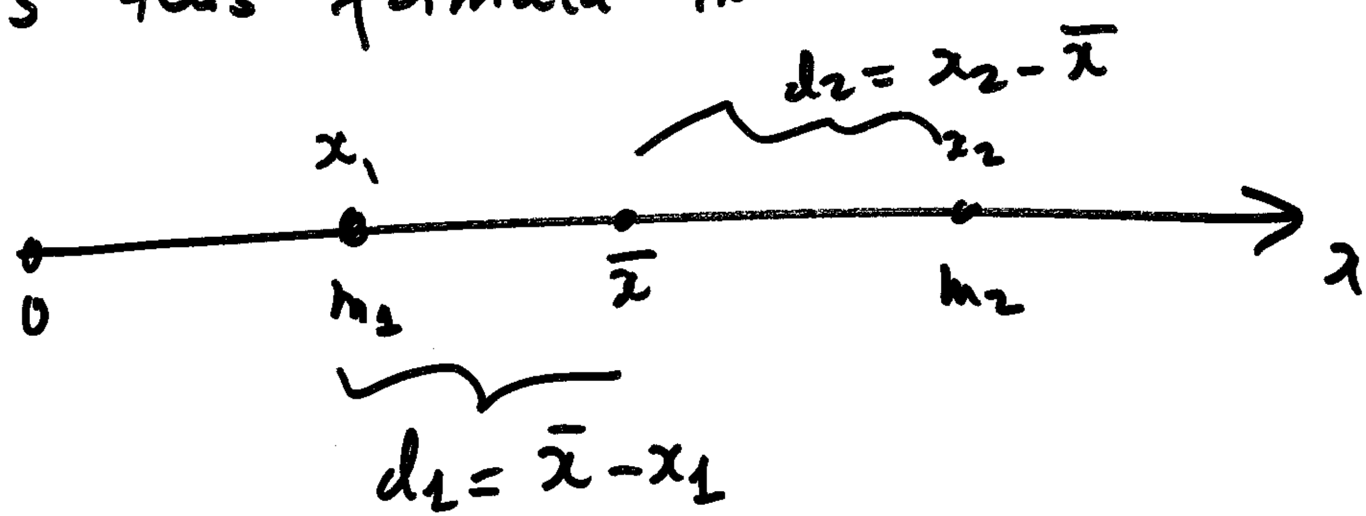
The center of mass is the point P where this plate balances horizontally

Two point masses



The Law of the lever: $m_1 d_1 = m_2 d_2$.

Express this formula in coordinates.



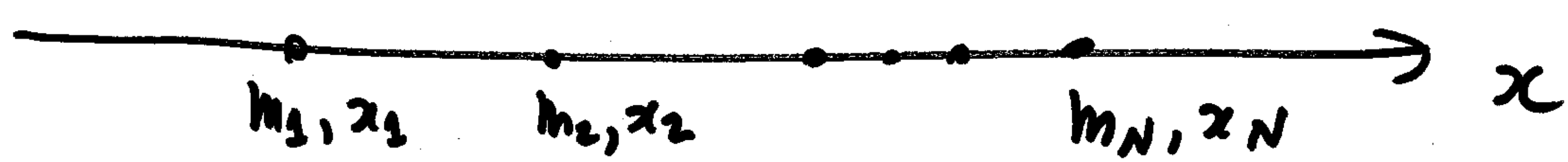
$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} - m_1 x_1 = m_2 x_2 - m_2 \bar{x}$$

$$m_1\bar{x} + m_2\bar{x} = m_1 x_1 + m_2 x_2.$$

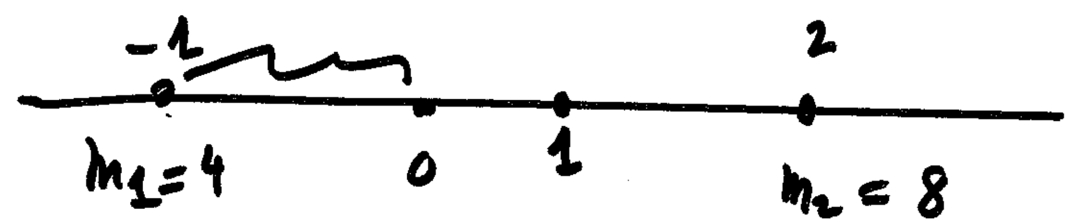
$$\bar{x}(m_1 + m_2) = m_1 x_1 + m_2 x_2.$$

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

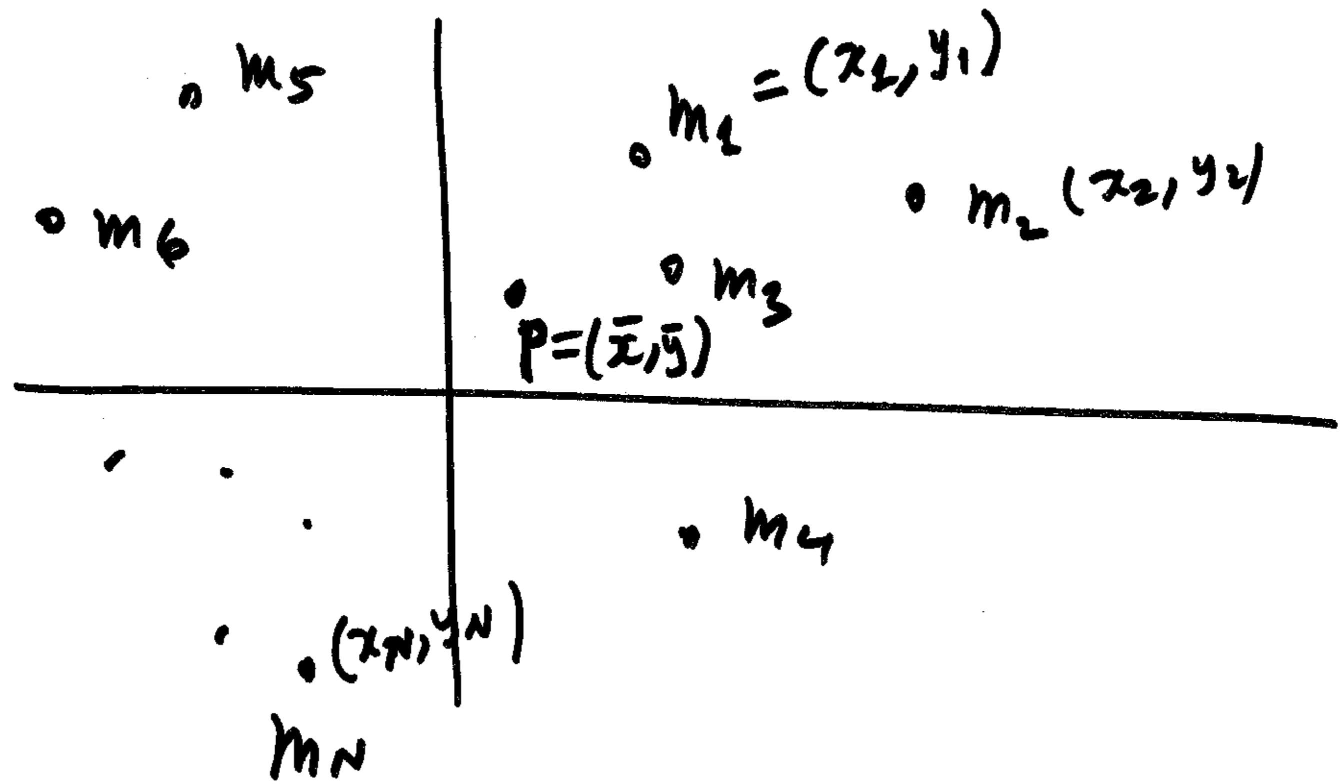


$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_N x_N}{m_1 + m_2 + \dots + m_N}.$$

<u>Ex:</u>

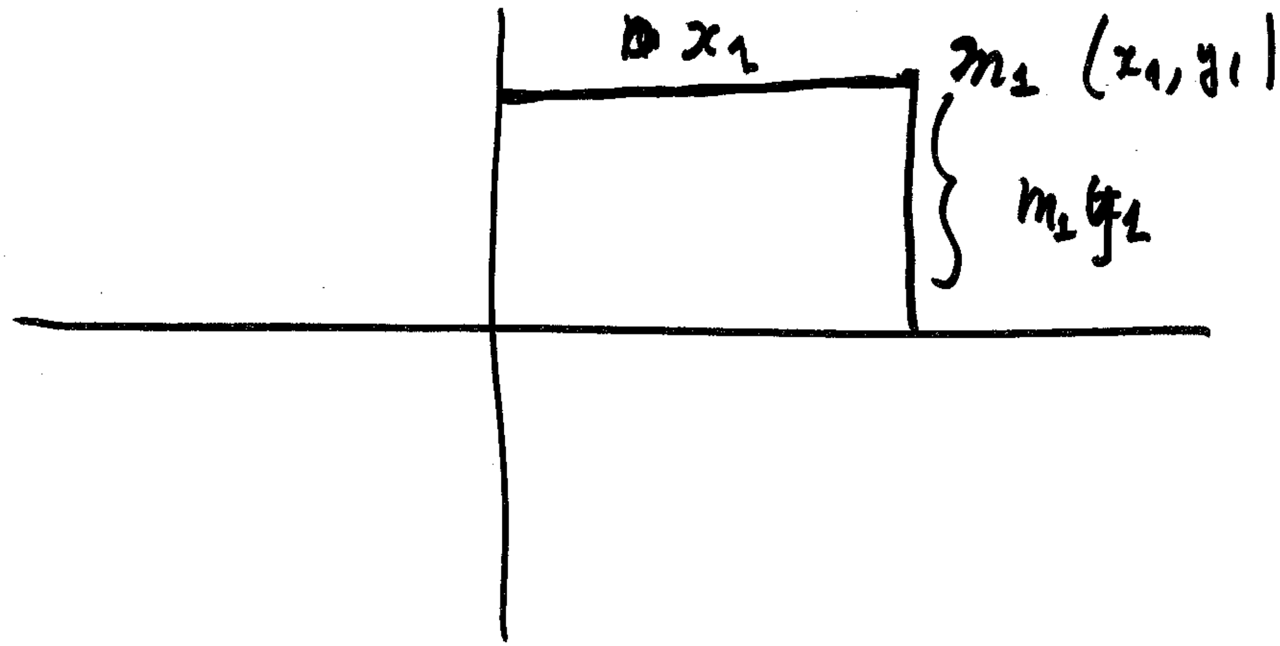


$$\bar{x} = \frac{4\cdot(-1) + 8(2)}{4+8} = \frac{16-4}{12} = 1$$



$$\bar{x} = \frac{x_1 m_1 + x_2 m_2 + \cdots + x_N m_N}{m_1 + m_2 + \cdots + m_N}$$

$$\bar{y} = \frac{y_1 m_1 + y_2 m_2 + \cdots + y_N m_N}{m_1 + m_2 + \cdots + m_N}$$



$m_1 x_1 =$ Moment of the mass m_1 with respect to the y-axis

$M_y = m_1 x_1 + m_2 x_2 + \cdots + m_N x_N$

$=$ Moment of the system with respect to the y-axis.

$m_1 y_1$ = moment of the mass m_1 with respect to the x-axis

$$M_x = m_1 y_1 + m_2 y_2 + \cdots + m_N y_N$$

= Total moment.

$$\bar{x} = \frac{M_y}{M} \quad ; \quad \bar{y} = \frac{M_x}{M}.$$

$$M = m_1 + m_2 + \cdots + m_N$$

Find the center of mass of the system

$$m_1 = \underline{\underline{5}}\ ;\ P_1(\underline{\underline{1}}, 2)$$

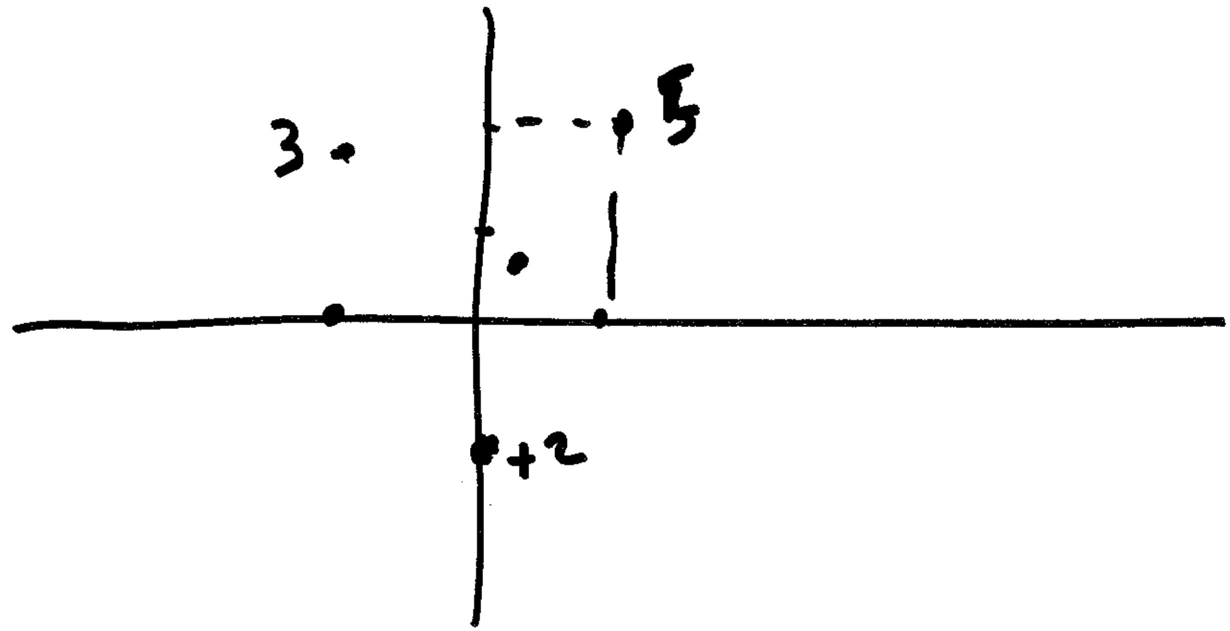
$$m_2 = 3\ ,\ P_2(-1, 4)$$

$$m_3 = 2\ ,\ P_3(0, -2)$$

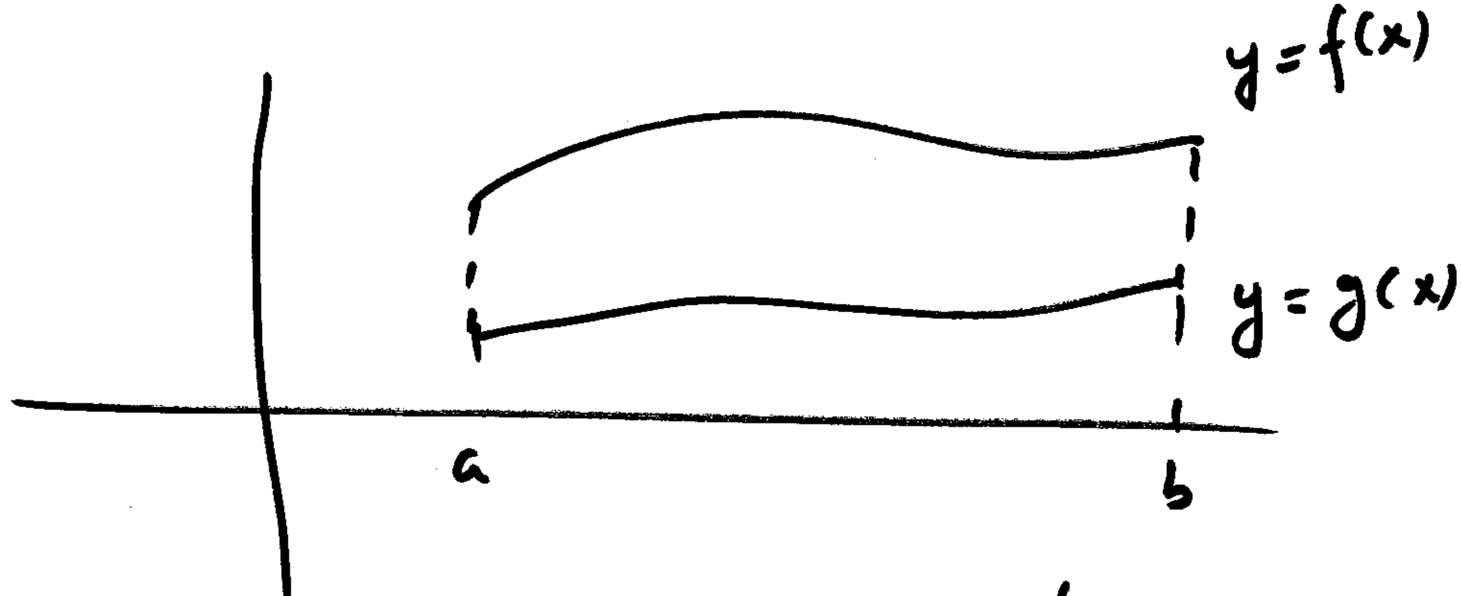
$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{5 + 3(-1) + 2 \cdot 0}{10}$$

$$= \frac{5-3}{10} = \frac{2}{10} = 1/5$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{10 + 12 - 4}{10} = \frac{18}{10}.$$

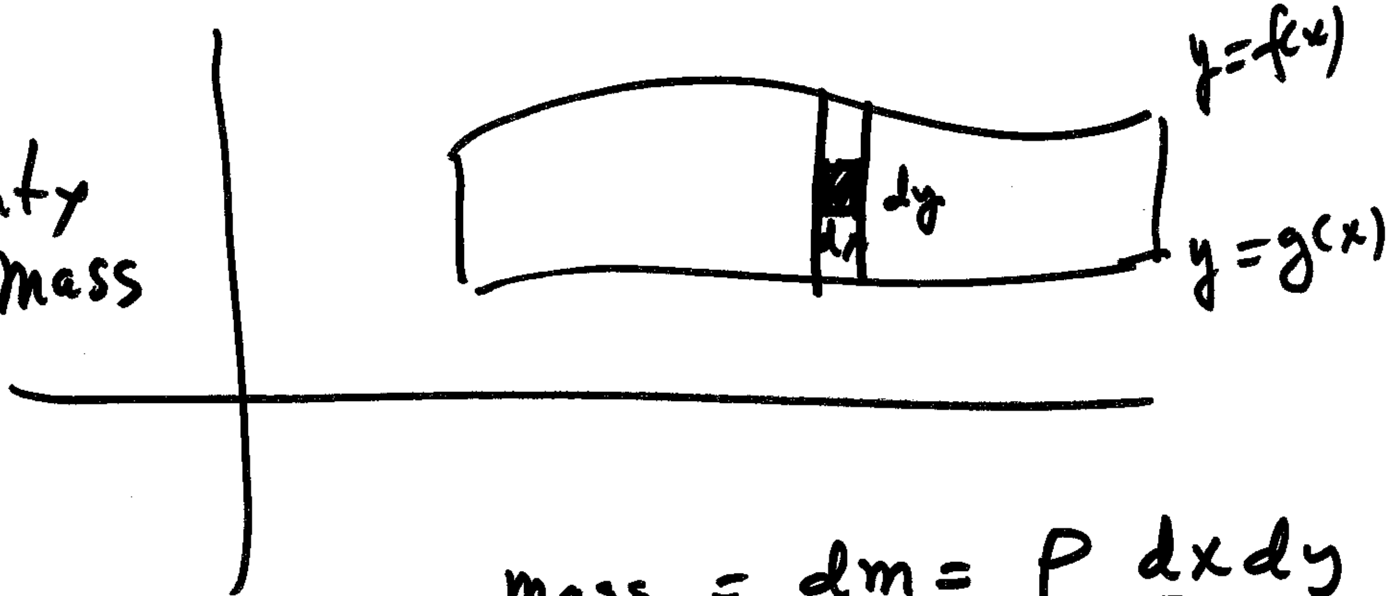

3
5
+2

Next we use integration to find the center of mass of a plate.



Assume that the plate is homogeneous: Density of mass is constant.

ρ = density of mass



mass $= dm = \rho \, \underline{\underline{dx}} \, \underline{\underline{dy}}$

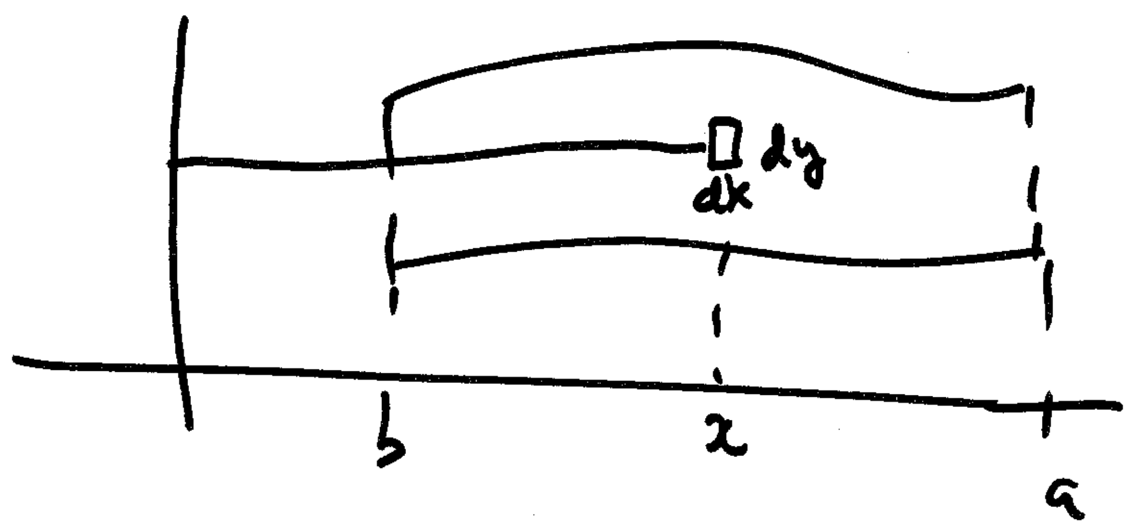
Integrate in y $\quad \displaystyle\int_{g(x)}^{f(x)} \rho \, dx \, dy$

$= \rho \left(f(x) - g(x) \right) dx$

<u>Integrate in x</u> $\quad M = \displaystyle\int_a^b \rho \left(f(x) - g(x) \right) dx.$

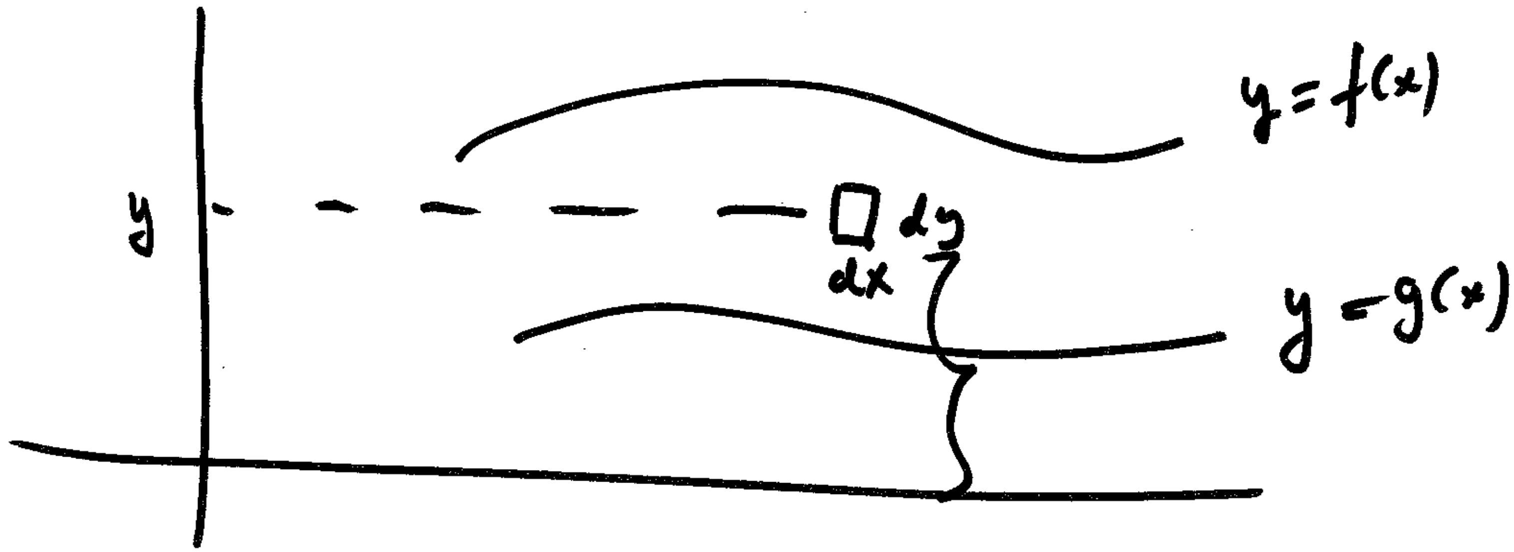
$$\boxed{M = \underline{\underline{\rho}} \int_a^b (f(x) - g(x))\, dx}$$

$\underline{\underline{M_y}}$



$$dM = \underbrace{x\ \rho}\, dx\, dy.$$

$$\boxed{M_y = \rho \int_a^b x\,(f(x) - g(x))\, dx}$$



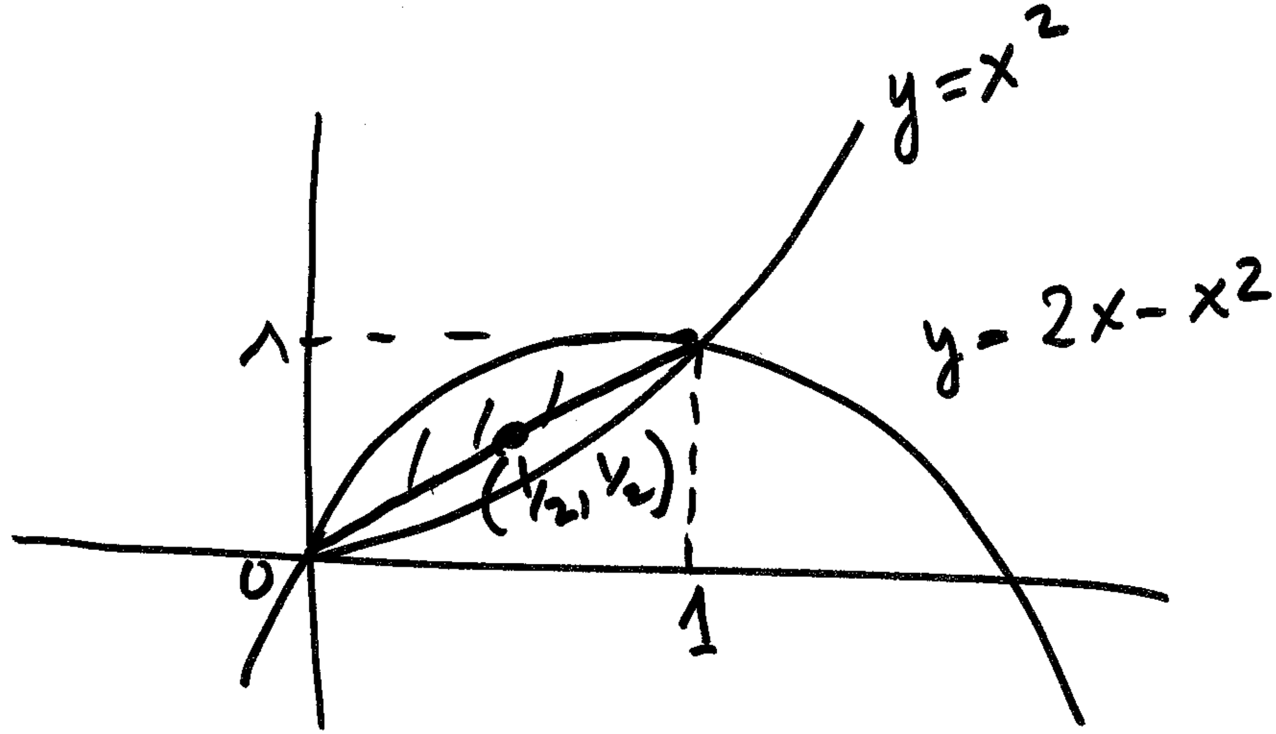
$$\cancel{M_x = \rho\, dx\, y\, dy}$$

$$dM_x = \rho\, \underline{\underline{y}}\, dx\, dy$$

$$\int_{g(x)}^{f(x)} y\, dy = \frac{1}{2} y^2 \Big|_{g(x)}^{f(x)} = \frac{1}{2}\left((f(x))^2 - (g(x))^2\right)$$

$$M_x = \frac{1}{2}\rho \int_a^b \left[\left(f(x)\right)^2 - \left(g(x)\right)^2 \right) dx.$$

$$\bar{x} = \frac{M_y}{oM} \; ; \quad \bar{y} = \frac{M_x}{M}.$$



Find the center of mass of the region bounded by the two curves.

$\rho = 1.$

$x^2 = 2x - x^2; \quad 2x^2 = 2x \qquad x=0$
$x=1$

$$M = \int_a^b (f(x) - g(x))\,dx \qquad f(x) = 2x - x^2$$
$$g(x) = x^2$$

$$M = \int_0^1 (2x - x^2 - x^2)\,dx$$
$$= \int_0^1 (2x - 2x^2)\,dx = 2\int_0^1 (x - x^2)\,dx$$
$$= 2.\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{1}{3}.$$

$$M_y = \int_0^1 x\,(f(x) - g(x))\,dx = \int_0^1 x\,(2x - 2x^2)\,dx$$
$$= 2\int_0^1 (x^2 - x^3)\,dx = 2\left(\tfrac{1}{3} - \tfrac{1}{4}\right) = \tfrac{1}{6}.$$

$$\overline{x} = \frac{M_y}{M}\ ; \quad \overline{x} = \frac{1/6}{1/3} = \tfrac{1}{2}.$$

$$M_x = \frac{1}{2}\int_0^1 \left[(f(x))^2 - (g(x))^2 \right] dx$$

$$= \frac{1}{2}\int_0^1 \left[(2x - x^2)^2 - (x^2)^2 \right] dx$$

$$= \frac{1}{2}\int_0^1 (4x^2 - 4x^3)\, dx = \dots$$

$$= 2\int_0^1 (x^2 - x^3) = 1/6$$

$$\bar{y} = \frac{1/6}{1/3} = 1/2.$$