Lesson 20 — Section 11.2 — Series

We begin with a sequence

$$\{a_1, a_2, \dots\} = \{a_n\}_{n=1}^{\infty}.$$

and define another sequence.

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + a_4 + \dots + a_n.$$

The question is whether

S_n converges.

$$\lim_{n \to \infty} S_n = S.$$

$$S_n = a_1 + a_2 + \dots + a_n = \sum_{i=1}^{n} a_i$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i$$ is defined to be an ~~i~~ series

Sometimes also called an infinite series.

The Geometric Series

$$\frac{1}{3} = 0.3333\ldots$$

What is the meaning of this?

$$0.3 = \frac{3}{10}$$

$$0.03 = \frac{3}{100} = \frac{3}{10^2}$$

$$0.003 = \frac{3}{10^3}$$

$$\boxed{\frac{1}{3} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots}$$

We want to understand the sum

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots + \frac{1}{10^n} + \cdots$$

The partial sums.

$$S_1 = \frac{1}{10}$$

$$S_2 = \frac{1}{10} + \frac{1}{10^2}$$

$$S_3 = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3}$$

$$\vdots$$

$$S_n = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n} = \sum_{i=1}^{n} \frac{1}{10^i}$$

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots = \boxed{\lim_{n\to\infty} S_n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{10^i}$$

$$= \sum_{i=1}^{\infty} \frac{1}{10^i}$$

In this case we can compute S_n.

$$S_n = \frac{1}{10} + \cancel{\frac{1}{10^2}} + \cancel{\frac{1}{10^3}} + \cdots + \frac{1}{10^{n-1}} + \cancel{\frac{1}{10^n}}$$

$$\frac{S_n}{10} = \cancel{\frac{1}{10^2}} + \cancel{\frac{1}{10^3}} + \frac{1}{10^4} + \cdots + \cancel{\frac{1}{10^n}} + \frac{1}{10^{n+1}}$$

$$\boxed{S_n - \frac{1}{10} S_n = \frac{1}{10} - \frac{1}{10^{n+1}}.}$$

$$S_n - \frac{1}{10} S_n = \frac{9}{10} S_n$$

$$\frac{9}{10} S_n = \frac{1}{10} - \frac{1}{10^{n+1}} \to 0 \text{ as } n \to \infty$$

$$\lim_{n\to\infty} \frac{9}{10} S_n = \frac{9}{10} \lim_{n\to\infty} S_n = \frac{1}{10}$$

Conclusion: $\frac{9}{10} \lim_{n\to\infty} S_n = \frac{1}{10}$

$$\boxed{\lim_{n\to\infty} S_n = \frac{1}{9}}$$

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots = \sum_{i=1}^{\infty} \frac{1}{10^i} = \frac{1}{9}.$$

If we multiply this by 3:

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots = \frac{3}{9} = \frac{1}{3}.$$

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots = \left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots$$

Geometric Series:

$$\sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + \cdots$$

where r is some real number

In the previous example, $r = \frac{1}{10}$.

<u>The partial sums:</u>

$$S_N = r + r^2 + r^3 + \cdots + r^{N-1} + r^N$$

$$r S_N = r^2 + r^3 + r^4 + \cdots + r^N + r^{N+1}.$$

$$S_N - r S_N = r - r^{N+1}.$$

$$(1-r) S_N = r - r^{N+1} \qquad r \neq 1.$$

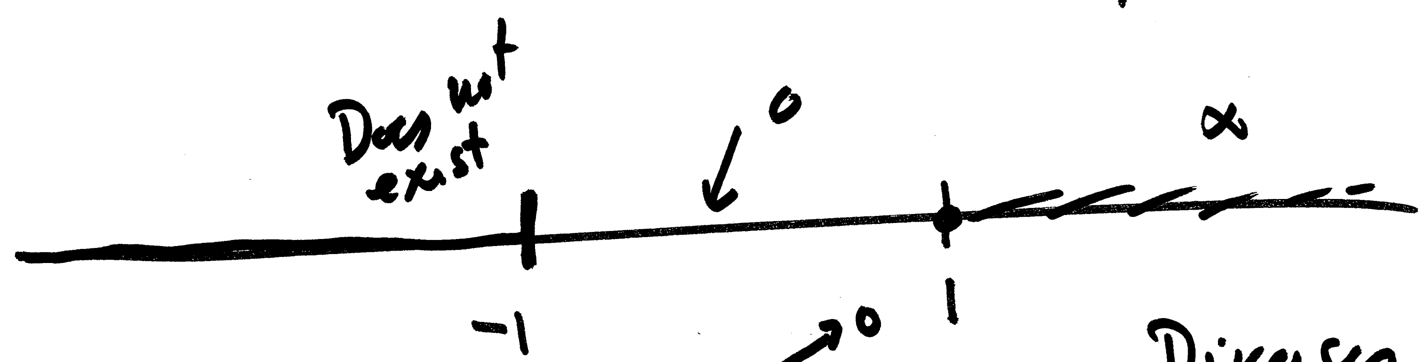
$$\boxed{S_N = \frac{r - r^{N+1}}{1-r}}$$

<u>when $r=1$</u>:

$$S_N = \underbrace{1+1+1+1+\cdots+1}_{N \text{ terms}} = N.$$

$$\lim_{N\to\infty} S_N = \infty \quad \text{if } r = 1$$

$$\lim_{N\to\infty} r^{N+1} = \begin{cases} 1 & \text{if } r=1 \\ =\infty & \text{if } r>1 \\ 0 & \text{if } |r|<1 \\ \text{Does not exist if} & r\le -1. \end{cases}$$



$$S_N = \frac{r - r^{N+1}}{1-r}.$$

$r^{N+1} \to 0$

Diverges otherwise.

$$\boxed{\lim_{N\to\infty} S_N = \frac{r}{1-r} \text{ if } |r|<1}$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = 1/2$$

$r = 1/3$

$$\boxed{\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots = \frac{1}{2}.}$$

Telescopic Series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \quad \sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}.$$

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Most of the time we are interested in finding out if the series converges or not. and one cannot compute the sum.

Test for divergence

If $\lim_{n\to\infty} a_n$ either does not exist

or if $\lim_{n\to\infty} a_n = L$, $L \neq 0$

then the series

$\sum_{n=1}^{\infty} a_n$ diverges.

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N$$

$$S_{N+1} = \sum_{n=1}^{N+1} a_n = a_1 + a_2 + \cdots + a_{N+1}$$

$$\boxed{S_{N+1} - S_N = a_{N+1}}$$

If the series converges $\lim_{N \to \infty} S_N = S$

then $\lim_{N \to \infty} S_{N+1} = S$

$$\boxed{\lim_{N \to \infty} a_{N+1} = 0 = \lim_{N \to \infty} a_N.}$$

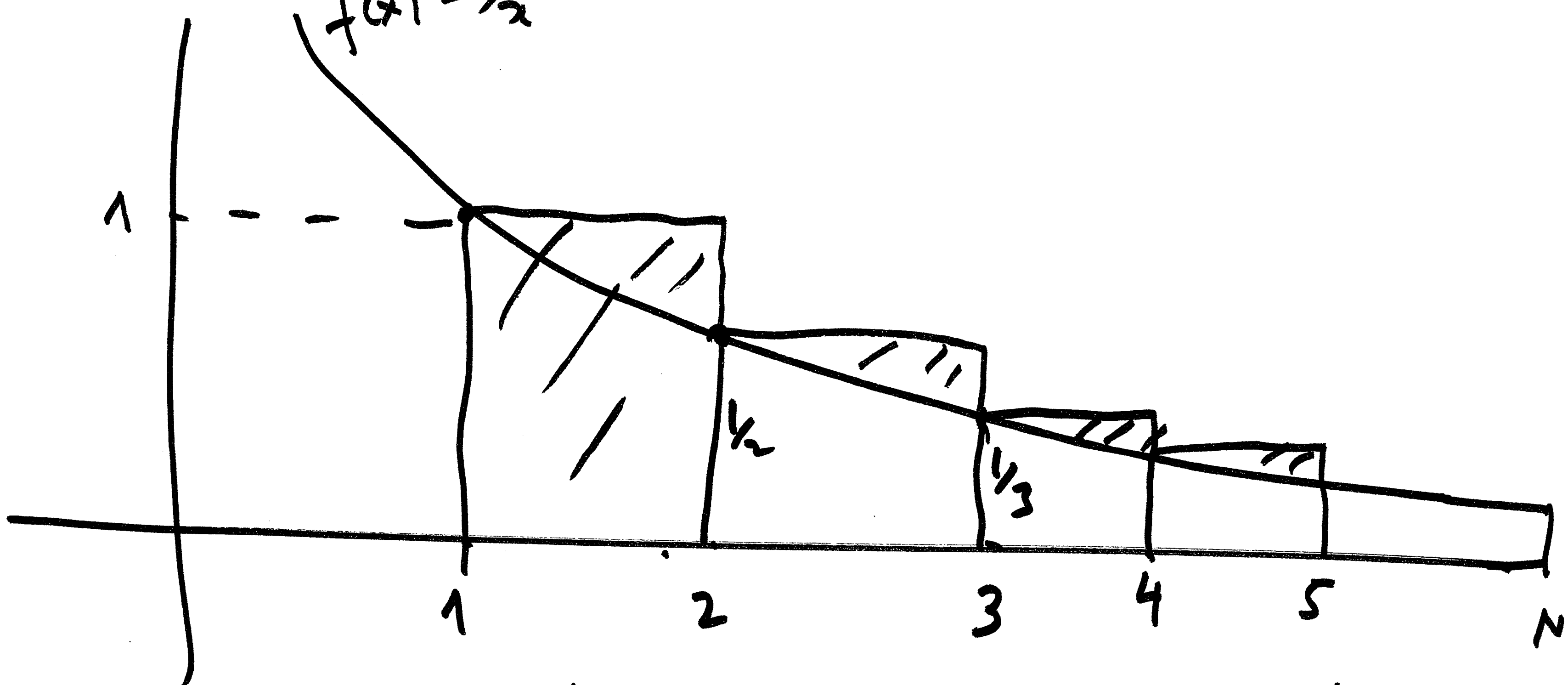
$$\sum_{n=1}^{\infty} \cos(1/n) \quad \text{Diverges.}$$

$$\lim_{n \to \infty} \cos(1/n) = 1 \neq 0$$

However if $\lim_{n \to \infty} a_n = 0$

one cannot say that

$$\sum_{n=1}^{\infty} a_n \quad \text{converges.}$$



$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{N}$$

$$\geq \int_1^N \frac{1}{x}\,dx = \ln N$$

$$S_N = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{N} \geq \log N$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \qquad a_n = \frac{1}{n(n+1)}$$

$$\lim_{n\to\infty} a_n = 0$$

However

$$\sum_{n=1}^{\infty} \frac{1}{n} \cdot \underline{\text{Diverges}} \qquad a_n = \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = 0$$