Lesson 21 **Section 11.3**

The Integral Test

Exam 2: Thursday, March 9.
8:00 - 9:00 PM Elliott

Lessons 9 to 21
With the exception of Lesson 15
(No trapezoid, Simpson's Rule, etc
No Table of Integrals).

The integral Test:

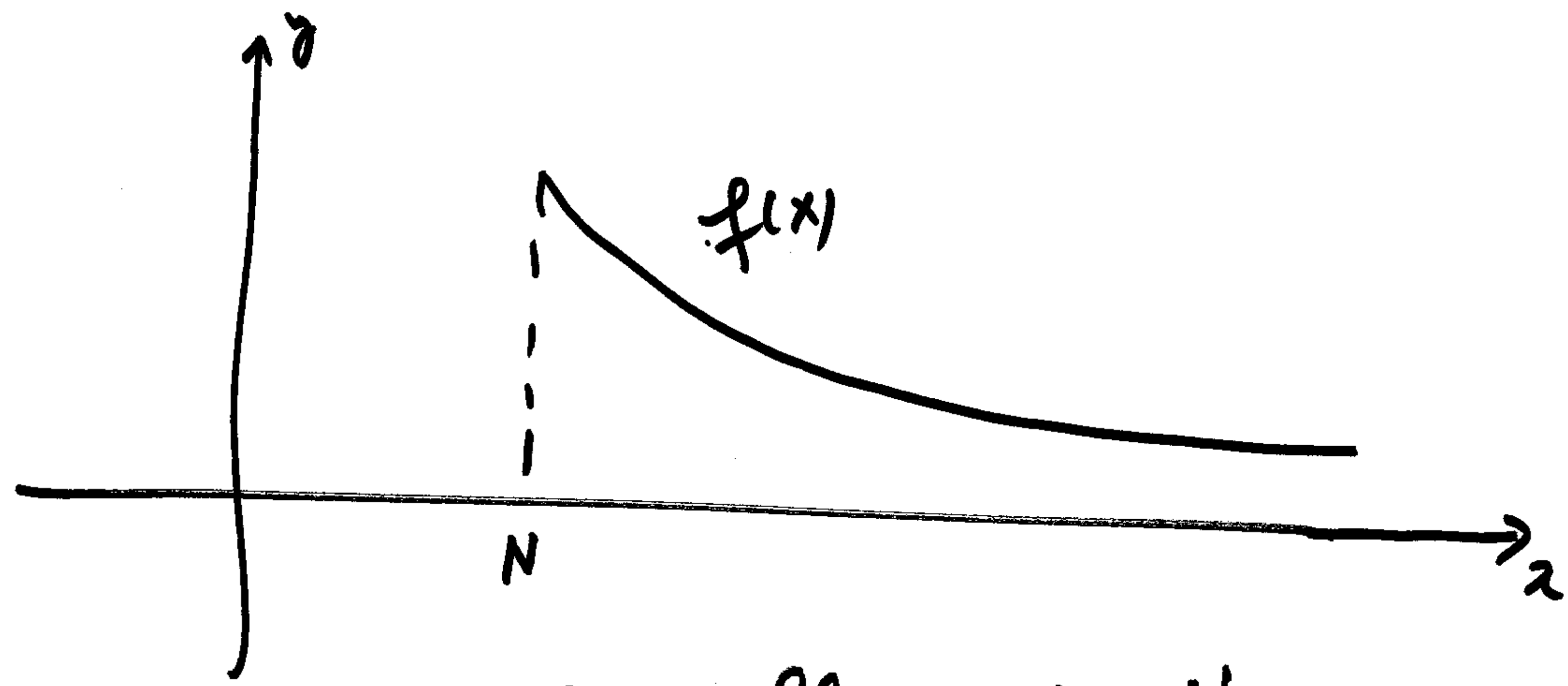
We study the convergence of series of the type

$$\sum_{n=1}^{\infty} f(n)$$

$f(x)$ is a function.

Ex.: $f(x) = \frac{1}{x^2}$, $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$f(x) = \frac{1}{x(\ln x)}$ $\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$

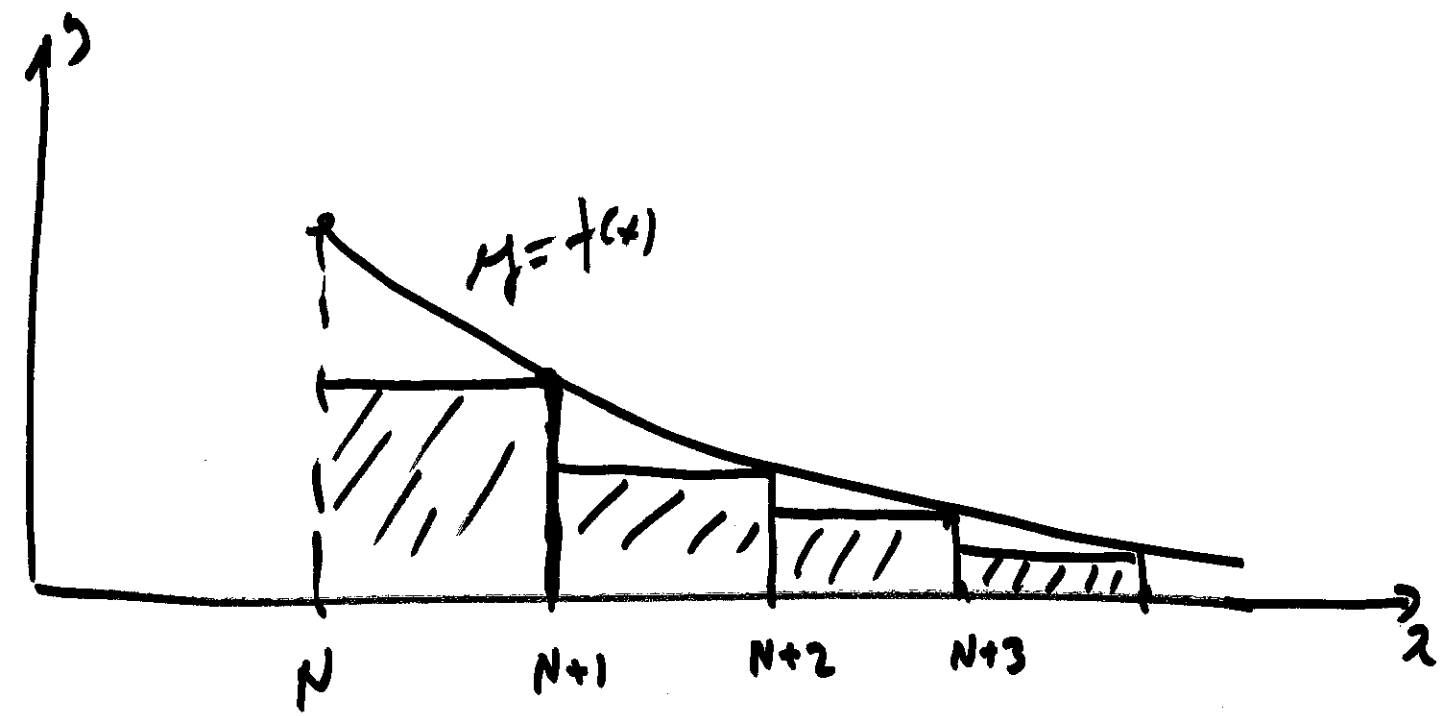


For all $\underline{x \geq N}$

(i) $f(x) > 0$

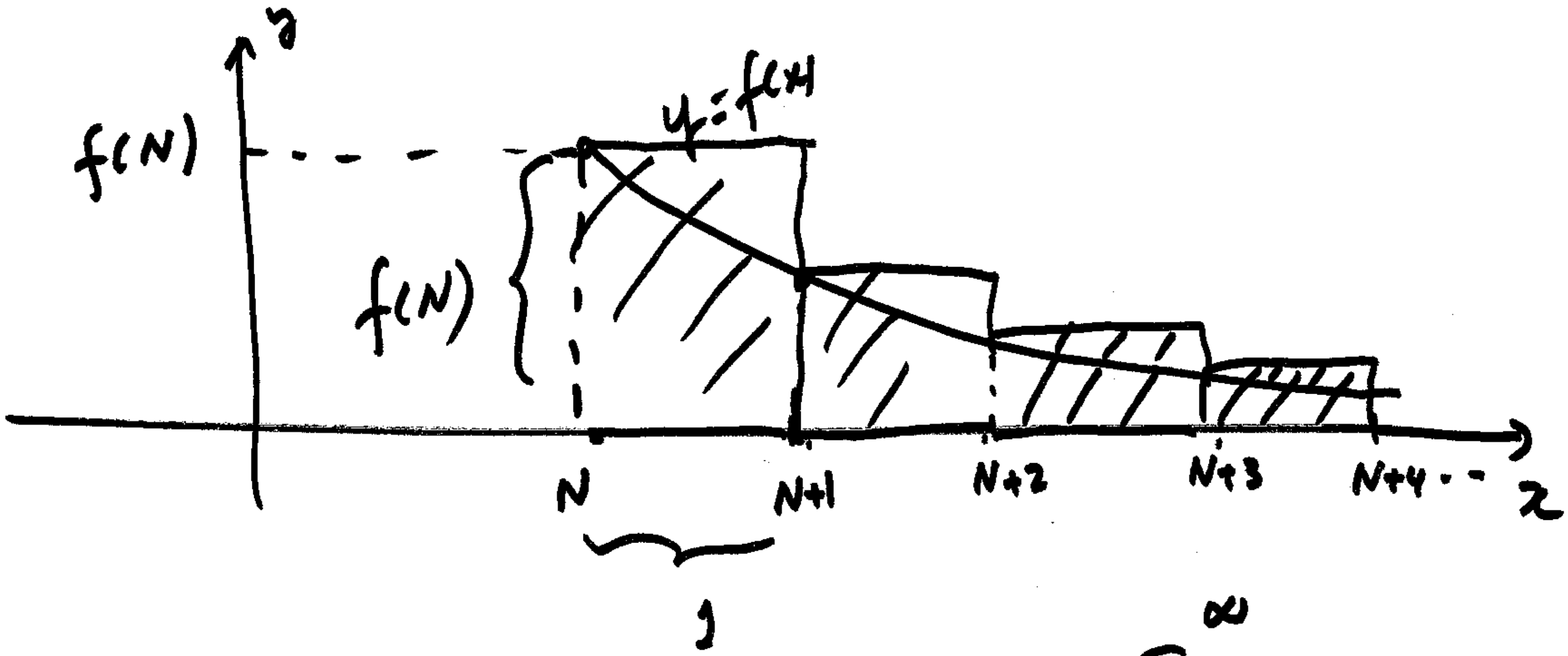
(ii) $f(x)$ is decreasing

(iii) $f(x)$ is continuous.



$$f(N+1) + f(N+2) + f(N+3) + f(N+4) + \cdots \leq \int_N^\infty f(x)\,dx$$

$$0 \leq \sum_{n=N+1}^{\infty} f(n) \leq \int_N^\infty f(x)\,dx.$$



$$f(N) + f(N+1) + f(N+3) + \dots \geqslant \int_N^\infty f(x)\,dx$$

$$\boxed{\sum_{n=N}^{\infty} f(n) \geqslant \int_N^\infty f(x)\,dx}$$

If $f(x)$ satisfies (i), (ii) and (iii)

and if

$$\int_N^\infty f(x)\,dx \text{ converges}$$

then

$$\sum_{n=1}^\infty f(n) \text{ converges}$$

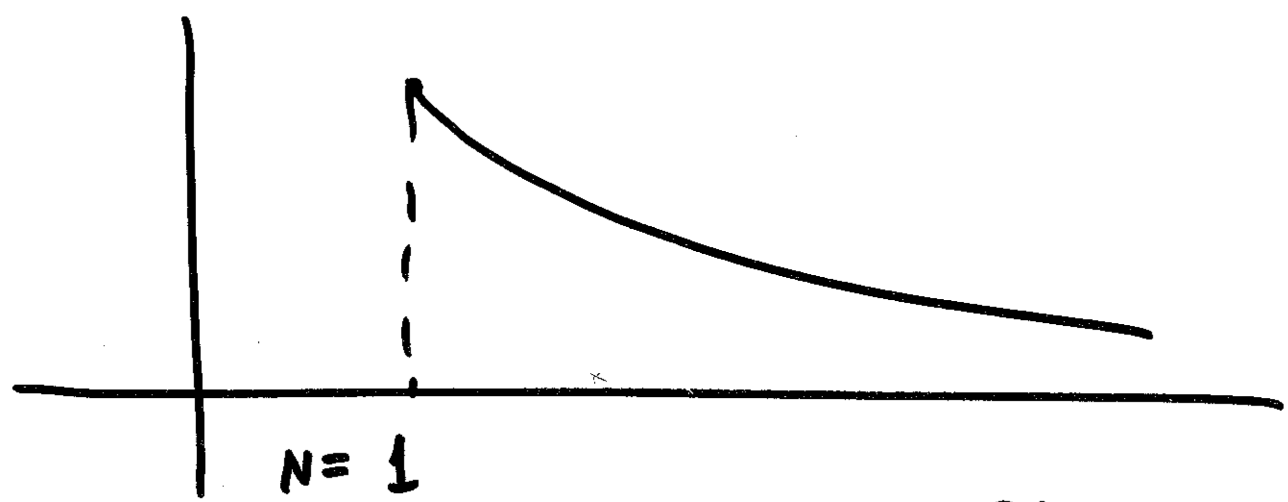
If

$$\int_N^\infty f(x)\,dx \text{ diverges.}$$

$$\sum_{n=1}^\infty f(n) \text{ diverges.}$$

<u>Examples:</u> $\sum_{n=1}^{\infty} \frac{1}{n}$ Diverges.

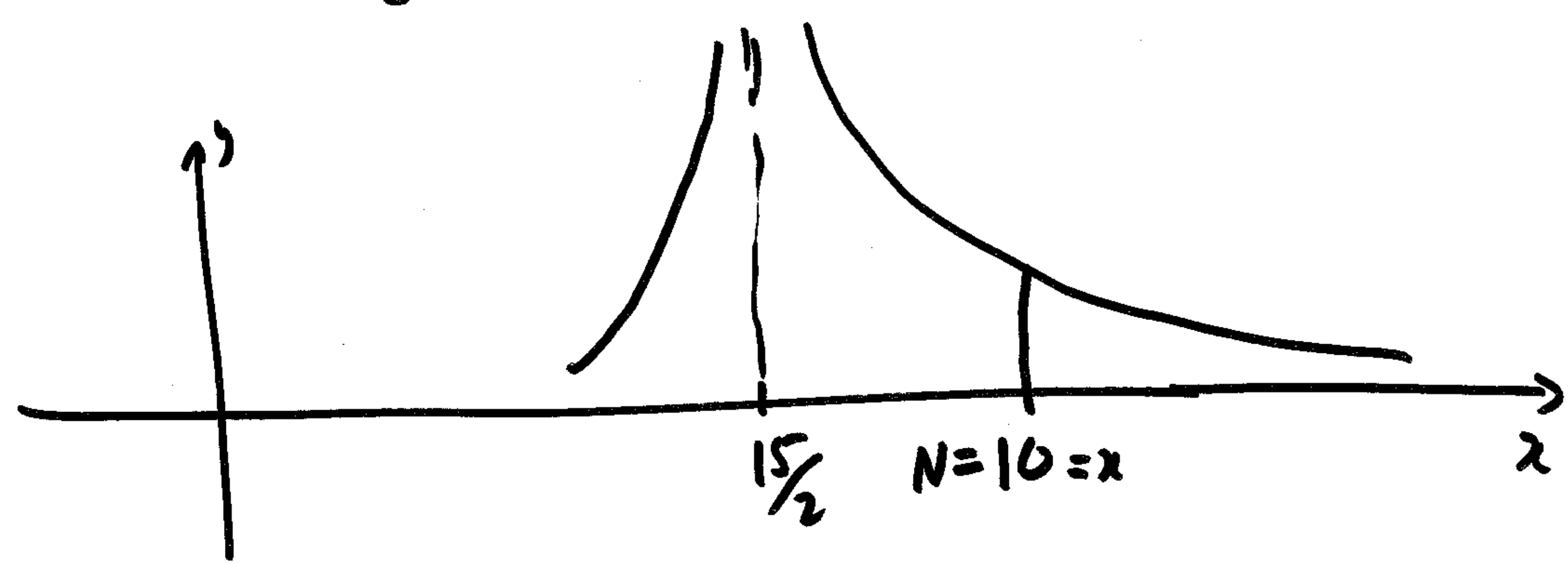
$f(x) = \frac{1}{x}$; $\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} f(n)$



$\int_1^{\infty} \frac{1}{x}\,dx = \ln x \Big|_1^{\infty}$ <u>diverges</u>.

$$\sum_{n=1}^{\infty} \frac{1}{\left(n-\frac{15}{2}\right)^2}$$

$$f(x)= \frac{1}{\left(x-\frac{15}{2}\right)^2}$$



$$\int_{10}^{\infty} \frac{1}{(x-15/2)^2}\,dx \qquad x-\frac{15}{2}=u$$

$$= \int_{5/2}^{\infty} \frac{1}{u^2}\,du = -u^{-1}\Big|_{5/2}^{\infty} = 2/5$$

Converges.

The integral test then says that

$$\sum_{n=1}^{\infty} \frac{1}{(n-15/2)^2}$$ converges.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

$$f(x) = \frac{1}{x(\ln x)^2} \; ; \quad x \geq 2$$

(i) $f(x) > 0$

(ii) Is $f(x)$ decreasing? Yes

(iii) $f(x)$ is continuous, Yes

$$f(x) = x^{-1}(\ln x)^{-2} \qquad \underline{x > 2}$$

$$f'(x) = -x^{-2}(\ln x)^{-2} - 2x^{-1}(\ln x)^{-3} \cdot x^{-1}$$

$$= -x^{-2}(\ln x)^{-2} - 2x^{-2}(\ln x)^{-3} \; \underline{< 0}$$

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx \quad , \quad u = \ln x, \quad du = \frac{1}{x}\,dx$$

$$= \int_{\ln 2}^\infty \frac{du}{u^2} = -\frac{1}{u}\Big|_{\ln 2}^\infty = (\ln 2)^{-1}$$

converges.

$$\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$$

converges!

P- Series

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \underline{p>0}}$$

Converges if $p>1$

Diverges if $p \leq 1$

$p = 1, 2, 3, .4, 5,$

$p = 0.2, \; 0.5, \; 0.003$

$$f(x) = \frac{1}{x^p} \qquad x \geq 1$$

$$f(x) = x^{-p} \; ; \quad f'(x) = -p\, x^{-p-1} < 0$$

We already saw test when $p=1$, thus diverges.

$\underline{p \neq 1}$:

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx$$

$$= \left.\frac{x^{1-p}}{1-p}\right|_1^\infty$$

When $\underline{p < 1}$, $\int_1^\infty \frac{1}{x^p}\,dx$ diverges

$p > 1$, $\int_1^\infty \frac{1}{x^p}\,dx$ <u>converges</u>

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$ Converges if $p > 1$
diverges if $p \leq 1$

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_{\ln 2}^{\infty} \frac{du}{u^p}$$

$u = \ln x$

$du = \frac{dx}{x}$

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.989}} \qquad p = 0.989 < 1$$

Diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.003}} \qquad p = 1.003 > 1$$

Converges.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{0.8}}$$ Diverges

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.5}}$$ Converges.

$$\sum_{n=1}^{\infty} e^{-n}$$ Converges.

$f(x) = e^{-x}$

$$\int_1^{\infty} e^{-x}\,dx = -e^{-x}\Big|_1^{\infty}$$

$$= 1/e.$$

Diverges.

$$\boxed{\sum_{n=1}^{\infty} \frac{n}{n^2+1}.}$$

$f(x) = \frac{x}{x^2+1}.$

$u = x^2+1$

$du = 2x\,dx$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \int_2^{\infty} \frac{du}{2u}. \text{ Div}$$