Lesson 22   Section 11.4 (Not on Exam 2)

The Comparison Tests

So far we know that

(L) $$\begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} & \text{Converges if } p > 1 \\ & \text{Diverges if } p \leq 1 \\[2ex] \displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} & \text{Converges if } p > 1 \\ & \text{Diverges if } p \leq 1. \\[2ex] \displaystyle\sum_{n=1}^{\infty} r^n & \text{Converges if } |r| < 1. \text{ and} \\ & \text{Diverges if } |r| \geq 1. \end{cases}$$

__Question:__ Analyze the convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{1+n}}{2+n} \; ; \qquad \sum_{n=1}^{\infty} \frac{n+2}{(n+1)^3}$$

$$\sum_{n=1}^{\infty} \frac{6^n}{5^n-1} \, .$$

We want to find a way of comparing these series to the ones in (L)

The comparison test

(1) We are given two series

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

$a_n \geq 0$ and $b_n \geq 0$ for all $n \geq N$

(2) $a_n \geq C\, b_n$, $C > 0$

for all $n \geq N$.

Conclusion: If $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} b_n$ also converges.

If $\sum_{n=1}^{\infty} b_n$ diverges, then

$\sum_{n=1}^{\infty} a_n$ also diverges.

<u>Example:</u> $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+5n^2+2n+8}$

$$n^3 + \underbrace{5n^2+2n+8} \geq n^3$$

$$\frac{1}{n^3+5n^2+2n+8} \leq \frac{1}{n^3}.$$

But we know that

$$\sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$$ diverges because $6/5 > 1$.

So $$\sum_{n=2}^{\infty} \frac{6^n}{5^n - 1}$$ also diverges.

But we already know that

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converges}$$

So $$\sum_{n=1}^{\infty} \frac{1}{n^3+5n^2+2n+8} \text{ converges.}$$

$$\sum_{n=2}^{\infty} \frac{6^n}{5^n-1}.$$

$$5^n - 1 \leq 5^n$$

$$\frac{1}{5^n-1} > \frac{1}{5^n}$$

$$\underline{\frac{6^n}{5^n-1}} > \frac{6^n}{5^n} = \left(\frac{6}{5}\right)^n$$

Suppose we have two series

$$\sum_{n=1}^{\infty} a_n \quad , \quad \sum_{n=1}^{\infty} b_n$$

(i) $a_n > 0$ and $b_n > 0$ for $n \geq N$

(ii) $0 < \overbrace{C_1 \leq \frac{a_n}{b_n}} \leq C_2$ for $n \geq N$.

Then Either

(I) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ Converge

OR

(II) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge.

$\underline{a_n \geq c_1 b_n}$, $a_n \leq c_2 b_n$

If $\sum b_n$ diverges, $\sum a_n$ diverges.

If $\sum a_n$ diverges, $\sum b_n$ diverges.

If $\sum_{n=1}^{\infty} a_n$ converges $\sum_{n=1}^{\infty} b_n$ converges

If $\sum_{n=1}^{\infty} b_n$ converges $\sum_{n=1}^{\infty} a_n$ converges.

The Limit Comparison theorem

$$\sum_{n=1}^{\infty} a_n \quad , \quad \sum_{n=1}^{\infty} b_n$$

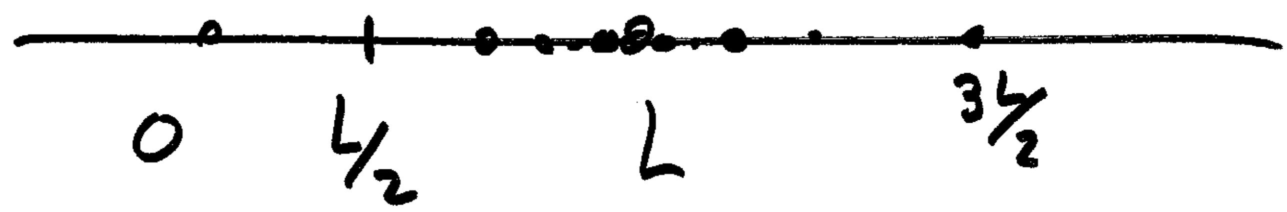
$a_n > 0$, $b_n > 0$ for $n \geq N$

If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$

then either both series converge

or both series diverge.

If $L>0$



$$\boxed{\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2} \quad \text{for } n \geq N}$$

$$\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1} = \sum_{n=1}^{\infty} \frac{6^n}{5^n(1-5^{-n})}$$

$$= \sum_{n=1}^{\infty} \boxed{\boxed{\left(\frac{6}{5}\right)^n} \cdot \frac{1}{1-5^{-n}}}$$

$$\frac{\frac{6^n}{5^n(1-5^{-n})}}{\left(6/5\right)^n} = \frac{1}{1-5^{-n}}$$

$$\frac{6^n}{5^n-1} \sim \frac{6^n}{5^n} \quad \text{as} \quad n \to \infty$$

$$\sum_{n=2}^{\infty} \frac{6^n}{5^n-1} \quad \text{and} \quad \sum_{n=2}^{\infty} \left(\frac{6}{5}\right)^n$$

converge or diverge together.

Since $\sum_{n=2}^{\infty} \left(\frac{6}{5}\right)^n$ diverges, the other one diverges as well.

14)

$$\sum_{n=1}^{\infty} \frac{1}{(3n^4+1)^{1/3}} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{4/3}(3+1/n^4)^{1/3}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{4/3}} \cdot \frac{1}{(3+1/n^4)^{1/3}}.$$

$$\frac{1}{(3n^4+1)^{1/3}} = \frac{1}{n^{4/3}} \cdot \frac{1}{(3+1/n^4)^{1/3}}$$

$$\frac{\dfrac{1}{(3n^4+1)^{1/3}}}{\dfrac{1}{n^{4/3}}} = \frac{1}{\left(3+\frac{1}{n^4}\right)^{1/3}} \longrightarrow \frac{1}{3^{1/3}} > 0$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n^4+1)^{1/3}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$

converge or diverge together.

But $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ converges, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n^4+1)^{1/3}}$

also converges.

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \cdot e^{1/n}$$

$$\frac{\frac{e^{1/n}}{n}}{1/n} = e^{1/n} \longrightarrow 1 \text{ as } n \to \infty$$

$$\sum_{n=1}^{\infty} \sin\left(1/n\right).$$

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$1/n = x$

$$\sum_{n=1}^{\infty} \sin 1/n \quad \text{and} \quad \sum_{n=1}^{\infty} 1/n \quad \text{diverge.}$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right).$$

$$\lim_{n\to\infty} \frac{\sin(1/n^3)}{1/n^3} \quad y = 1/n^3$$

$$= \lim_{y\to 0} \frac{\sin y}{y} = 1$$

Both converge!

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^3}$$