MA 162 : Section 11.6 Lesson 24

Absolute convergence, the Ratio and Root tests.

Example 1: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges.

$\displaystyle\frac{1}{n^2} = \left| \frac{(-1)^n}{n^2} \right|$ Sign does not matter.

However

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{converges}$$

but

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges.}$$

Here the sign matters!

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + 1/2 - 1/3 + 1/4 - 1/5 + 1/6$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + 1/2 + 1/3 + 1/4 + 1/5 + \cdots$$

A Series $\sum_{n=1}^{\infty} a_n$ converges absolutely

if $\sum_{n=1}^{\infty} |a_n|$ converges.

The examples we just saw:

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely

But

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not.

Remark: If $\sum_{n=1}^{\infty} |a_n|$ converges

then $\sum_{n=1}^{\infty} a_n$ also converges.

Here is why: $0 \leq a_n + |a_n| \leq 2|a_n|$

If $\sum_{n=1}^{\infty} |a_n|$ converges, then

by the comparison theorem $\sum_{n=1}^{\infty} (a_n + |a_n|)$

also converges.

$$a_n = (a_n + |a_n|) - |a_n|$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

So $\sum_{n=1}^{\infty} a_n$ converges as well.

However, as we saw $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}$ converges but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not.

If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge

we say that $\sum_{n=1}^{\infty} a_n$ is

conditionally convergent.

Tests For absolute Convergence

We are given a Series $\sum_{n=1}^{\infty} a_n$

Question: Does $\sum_{n=1}^{\infty} |a_n|$ converge?

Test: (The Ratio test)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

(i) If $L < 1$, then $\sum_{n=1}^{\infty} |a_n|$ converges.

(ii) If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges

(iii) If $L = 1$. The test is no good.

<u>Examples with $L=1$</u>

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad a_n = \frac{1}{n}$$

$$a_{n+1} = \frac{1}{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{1/_{n+1}}{1/_n} = \frac{n}{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{Converges.}$$

$$a_{n+1} = \frac{1}{(n+1)^2}, \quad a_n = \frac{1}{n^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2$$

$$\boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1}$$

$$\sum_{n=1}^{\infty} \frac{n}{5^n}$$

$$a_n = \frac{n}{5^n}$$

$$a_{n+1} = \frac{n+1}{5^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{5^{n+1}} \times \frac{5^n}{n} = \frac{5^n}{5^{n+1}} \cdot \frac{n+1}{n}$$

$$= \frac{1}{5} \frac{n+1}{n}.$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{5} < 1.$$ So

the Series converges!

$$\boxed{\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}}$$

Diverges!

$$a_n = \frac{(-2)^n}{n^2}$$

$$|a_n| = \frac{2^n}{n^2}; \quad |a_{n+1}| = \frac{2^{n+1}}{(n+1)^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} = 2 \cdot \frac{n^2}{(n+1)^2}.$$

$$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = 2 > 1.$$

Diverges.

The Root test: Given a series

$$\sum_{n=1}^{\infty} a_n .$$

$$L = \lim_{n \to \infty} |a_n|^{1/n}$$

(i) If $L < 1$, $\sum_{n=1}^{\infty} |a_n|$ converges

(ii) If $L > 1$ or $L = \infty$, $\sum_{n=1}^{\infty} a_n$ diverges

(iii) If $L = 1$, the test is inconclusive.

$$\sum_{n=1}^{\infty} \left(2 + 1/n\right)^{n}$$

$$a_n = \left(2 + 1/n\right)^{n}$$

$$a_n^{1/n} = 2 + 1/n.$$

$$\lim_{n\to\infty} a_n^{1/n} = 2 > 1.$$

So the series diverges!

$$\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n}\right)^{n}$$ Converges absolutely.

When n is very large $\left(\frac{1-n}{2+3n}\right)^{n} \sim \left(-\frac{1}{3}\right)^{n}$

Apply the test: $a_n = \left(\frac{1-n}{2+3n}\right)^{n}$

$$|a_n| = \left(\frac{n-1}{2+3n}\right)^{n} \qquad |a_n|^{1/n} = \frac{n-1}{2+3n}$$

$$\lim_{n\to\infty} |a_n|^{1/n} = 1/3 < 1.$$

$$\sum_{n=1}^{\infty} (\arctan n)^n$$

$$a_n = (\arctan n)^n$$

$$(a_n)^{1/n} = \arctan n$$

$$\lim_{n \to \infty} (a_n)^{1/n} = \pi/2 > 1.$$ Diverges.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = L$$

$$\ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right) = \ln L$$

$$\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n =$$

$$= \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) =$$

$$= \lim_{n\to\infty}\frac{\ln\left(1+1/n\right)}{1/n}$$

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$$ Diverges.

$$a_n = \left(1+\frac{1}{n}\right)^{n^2}$$

$$(a_n)^{1/n} = \left(1+\frac{1}{n}\right)^{n}$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e > 1$$

$$= \lim_{n \to \infty} \frac{\frac{1}{1+1/n} \cdot -1/n^2}{-1/n^2}$$

$$= \lim_{n \to \infty} \frac{1}{1+1/n} = 1.$$

<u>Conclusion:</u> $\ln\left(\lim_{n \to \infty} \left(1 + 1/n\right)^n \right) = 1$

$$\boxed{\lim_{n \to \infty} \left(1 + a/n\right)^n = e^a}$$