Lesson 25 Review of what we have done so far on series.

We have a series $\sum_{n=1}^{\infty} a_n$.

and we want to know if it converges.

Test 1 (Test of divergence) Take $\lim_{n\to\infty} a_n$.. If it does not exist or is not equal to zero $\sum_{n=1}^{\infty} a_n$ diverges.

However, if $\lim_{n \to \infty} a_n = 0$ the series might converge or diverge.

<u>Examples where this is useful</u>:

1) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{5n+3}$ $\qquad a_n = \dfrac{n+1}{5n+3}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n+1}{5n+3} = 1/5 \neq 0.$$

Series diverges.

$$\sum_{n=1}^{\infty} e^{1/n} \qquad a_n = e^{1/n}$$

Diverges. $\lim_{n\to\infty} a_n = 1 \neq 0$

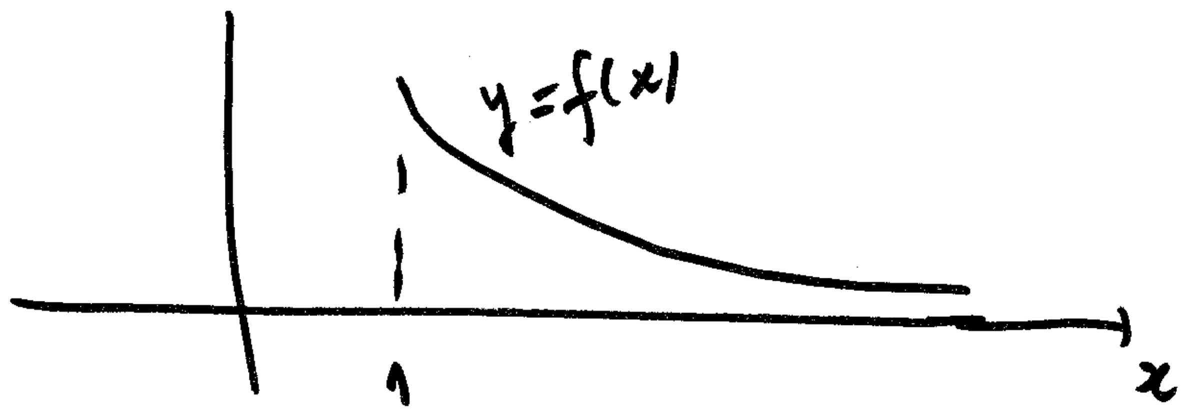
Examples where this test is not useful:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$a_n = 1/n$ or $a_n = 1/n^2$

The integral test:

If $f(x) \geq 0$; decreasing for $x \geq 1$ and $\lim_{x \to \infty} f(x) = 0$



The Series $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Examples where this test is useful

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ 2) $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

3) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$.

3) $f(x) = 1/x$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges if and only if

$$\int_1^{\infty} \frac{1}{x}\,dx \text{ converges.}$$

But $\int_1^\infty \frac{1}{x}\,dx$ diverges,

so

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$f(x) = \frac{1}{x^2}$$

$$\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = 1$$

Converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad f(x) = \frac{1}{x^p}$$

Converges if $p > 1$

Diverges if $p \leq 1$.

$$\sum_{n=1}^{\infty} e^{-n} \qquad f(x) = e^{-x}$$

$$\int_1^{\infty} e^{-x}\,dx = -e^{-x}\Big|_1^{\infty}$$

$$= e^{-1} \text{ finite}$$

Converses.

The Comparison tests.

We know that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

diverges if $p \leq 1$.

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if $p > 1$

diverges if $p \leq 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges

<u>The Comparison test.</u> If

$$0 \leq a_n \leq b_n$$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+10}$$

$$\frac{1}{n^2+3n+10} < \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+10}$ converges.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n+3}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3}$ diverges $\qquad n+3 > n$

$$\underline{\underline{\frac{1}{n+3}}} < \underline{\underline{\frac{1}{n}}}$$

$$n+3 < 3n$$

$$\frac{1}{n+3} > \frac{1}{3n}$$

But $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

But $\frac{\ln n}{n+3} > \frac{1}{n+3}$

So $\sum_{n=1}^{\infty} \frac{\ln n}{n+3}$ diverges.

<u>Refined Version</u>: The limit Comparison test.

$a_n > 0, \quad b_n > 0$

$\lim_{n\to\infty} \frac{a_n}{b_n} = L, \quad L \neq 0$

Then Both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge

or Both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge.

<u>Ex:</u> $$\sum_{n=1}^{\infty} \frac{n^2+5}{n^3+8}$$

$$\frac{n^2+5}{n^3+8} \sim \frac{n^2}{n^3} = 1/n \text{ when } n \text{ is large}$$

$$a_n = \frac{n^2+5}{n^3+8}, \quad b_n = \frac{1}{n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\sum_{n=1}^{\infty} \frac{n^2+5}{n^3+8}$ diverges.

$$\sum_{n=1}^{\infty} \sin\left(1/n\right) \quad ; \quad \sum_{n=1}^{\infty} \sin\left(1/n^2\right).$$

<u>Test:</u> $a_n = \sin\left(1/n\right)$; $b_n = 1/n$

$$\frac{a_n}{b_n} = \frac{\sin(1/n)}{1/n} = 1.$$

Set $y = 1/n$

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{y\to 0} \frac{\sin y}{y} = 1$$

$\sum_{n=1}^{\infty} \sin(1/n)$ and $\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$

Diverges.

$\sum_{n=1}^{\infty} \sin(1/n^2)$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converge

The Ratio and Root Tests

$$\sum_{n=1}^{\infty} a_n .$$

The Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

The Root test

$$\lim_{n\to\infty} |a_n|^{1/n} = L$$

$$\boxed{\begin{array}{lll} \text{If } L<1, & \sum |a_n| & \text{converges.} \\ \phantom{\text{If }} L>1 & \sum a_n & \text{diverges.} \\ \phantom{\text{If }} L=1 & \text{Nothing can be said.} & \end{array}}$$

Examples: (i) $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1 \right)^n$

(ii) $\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n} \right)^n$.

(i) $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1 \right)^n$

$$a_n = \left(\sqrt[n]{2} - 1 \right)^n$$

$$(a_n)^{1/n} = \sqrt[n]{2} - 1 = 2^{1/n} - 1.$$

$$\lim_{n\to\infty}\left(2^{1/n}-1\right) = 1-1 = 0 < 1.$$

Series converges.

$$\sum_{n=1}^{\infty}\left(\frac{1-n}{2+3n}\right)^n ; \quad \sum_{n=1}^{\infty}\left(\frac{3+5n}{8+3n}\right)^n .$$

$$a_n = \left(\frac{1-n}{2+3n}\right)^n, \quad |a_n| = \left(\frac{n-1}{2+3n}\right)^{1/n}$$

$$|a_n|^{1/n} = \frac{n-1}{2+3n} \to 1/3 < 1.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$ Compare to $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{1/n}{1/n^{1+1/n}} =$$

$$\lim_{n\to\infty} \frac{n^{1+1/n}}{n} = \lim_{n\to\infty} n^{1/n} = 1$$

$$\lim_{x \to \infty} x^{1/x} = 1.$$

$$\ln\left(\lim_{x \to \infty} x^{1/x}\right) = \lim_{x \to \infty} \ln\left(x^{1/x}\right)$$

$$= \lim_{x \to \infty} \frac{1}{x} \ln x = 0$$