Lesson 26 — Section 11.8

Power Series are Series of the following Types:

$$\sum_{n=1}^{\infty} c_n x^n \quad \text{Centered at } 0$$

or

$$\sum_{n=1}^{\infty} \underline{\underline{c_n}} (x-a)^n. \quad \text{Centered at } a$$

__Our goal:__ Use the ratio test to ~~sp~~ study the convergence of such series:

__Example:__ For what values of x does

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n\, n}$$

converge?

$$c_n = \frac{1}{3^n \cdot n} \qquad a = 2.$$

The series ~~obviously~~ obviously converges when $\underline{x = 2.}$

The ratio test: $\lim\limits_{n \to 0} \left| \dfrac{a_{n+1}}{a_n} \right|$

$$a_n = \frac{(x-2)^n}{3^n \cdot n} \qquad a_{n+1} = \frac{(x-2)^{n+1}}{3^{n+1} \cdot (n+1)}$$

$\underline{x \neq 2}$ $\quad \dfrac{a_{n+1}}{a_n} = \dfrac{(x-2)^{n+1}}{3^{n+1}(n+1)} \cdot \dfrac{3^n \cdot n}{(x-2)^n}.$

$$\frac{a_{n+1}}{a_n} = \frac{x-2}{3} \cdot \frac{n}{n+1}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x-2}{3}\right| \cdot \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \frac{|x-2|}{3}.$$

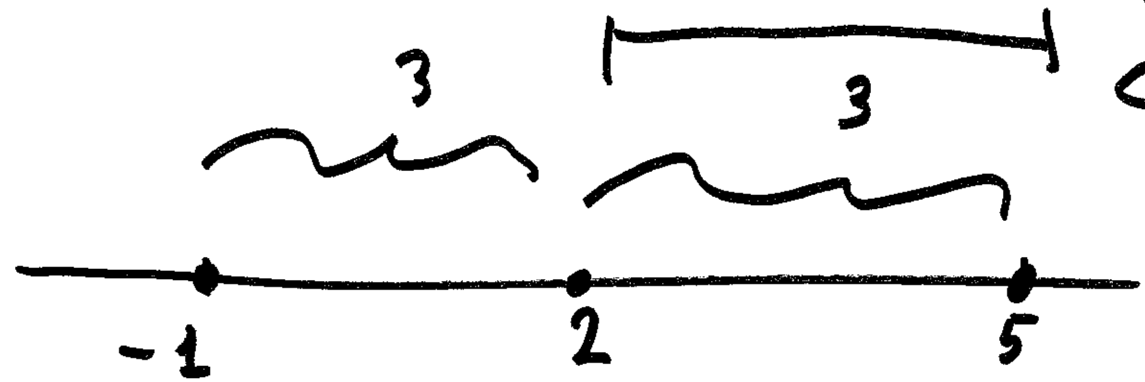
when $\frac{|x-2|}{3} < 1$, the series converges absolutely

$\frac{|x-2|}{3} > 1$, the series diverges.

$$\left|\frac{x-2}{3}\right| < 1$$

$$|x-2| < 3$$

3 = The Radius of convergence



When $-1 < x < 5$, the Series converges absolutely

when $x > 5$ or $x < -1$ the Series diverges.

<u>Conclusion:</u> The Series converges if $-1 \le x < 5$

$[-1, 5)$ = The interval of convergence.

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n \cdot n}$$

<u>When $x=5$</u>: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{3^n \cdot n} = \sum_{n=1}^{\infty} \frac{1}{n}$.

Diverges

When $x=-1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n \cdot n}$

$$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{3^n}{3^n \cdot n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Converges.

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2\, 4^n}$$

$$a_n = \frac{(x-1)^n}{n^2\, 4^n} \qquad a_{n+1} = \frac{(x-1)^{n+1}}{(n+1)^2 \cdot 4^{n+1}}$$

$\underline{x \neq 1}$

$$\frac{a_{n+1}}{a_n} = \frac{(x-1)^{n+1}}{(n+1)^2\, 4^{n+1}} \cdot \frac{n^2\, 4^n}{(x-1)^n} = \frac{x-1}{4} \frac{n^2}{(n+1)^2}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-1|}{4} \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= \frac{|x-1|}{4}.$$

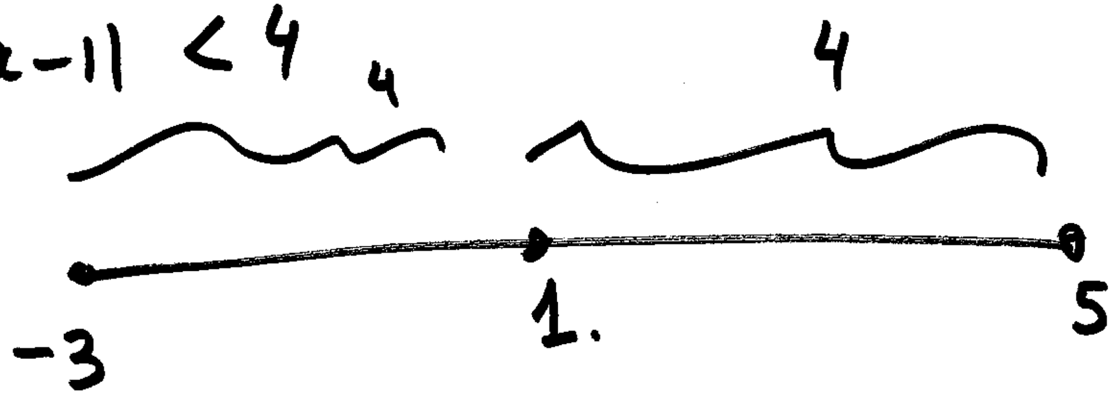
Thus series converges when

$$\frac{|x-1|}{4} < 1$$

$$\boxed{|x-1| < 4}$$ Radius of convergence.

Diverges when $|x-1| > 4$.

$|x-1| < 4$



$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2 4^n}$$

Check the convergence at $x = 5$ and $x = -3$

Separately:

When $\underline{x = 5}$: $$\sum_{n=1}^{\infty} \frac{4^n}{n^2 4^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$x = -3$ $$\sum_{n=1}^{\infty} \frac{(-4)^n}{n^2 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Interval of Convergence: $[-3, 5]$.

Example: $\sum_{n=1}^{\infty} n^n (x-1)^n$

Converges when $x=1$

$$a_n = n^n (x-1)^n = \left(n(x-1)\right)^n$$

$$|a_n|^{1/n} = |n(x-1)|$$

when $x \neq 1$ $\quad \lim_{n\to\infty} |n(x-1)| = \infty$

The series diverges for all $x \neq 1$.

Radius of Convergence **$R=0$**

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$a_n = \frac{x^n}{n!} \quad , \quad a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$x \neq 0$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = x \, \frac{n!}{(n+1)!}$$

$$n! = 1.\ 2.\ 3 \ldots .\ n$$

$$(n+1)! = 1.\ 2.\ 3 \ldots\ n.\ (n+1)$$

$$\frac{n!}{(n+1)!} = \frac{\cancel{1.\ 2.\ 3 \ldots}\ n}{1.\ \cancel{2.\ 3 \ldots}\ n.\ (n+1)} = \frac{1}{n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$$

This series converges for every x.

The interval of convergence is $(-\infty, \infty)$ the radius of convergence is ∞.

For an arbitrary power series

$$\sum_{n=1}^{\infty} c_n (x-a)^n$$

there are three possibilities for interval of convergence:

(i) It converges only for $x = a$ and the radius of convergence is

$$R = 0.$$

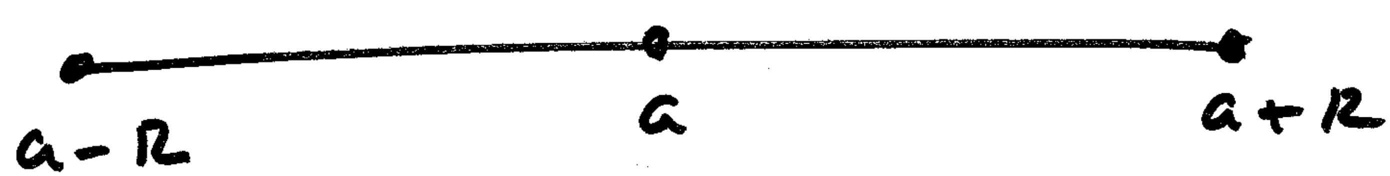
(ii) It converges for all x in $(-\infty, \infty)$ the radius of convergence is

$$R = \infty.$$

(iii) It converges for x on an interval

$$|x-a| < R.$$

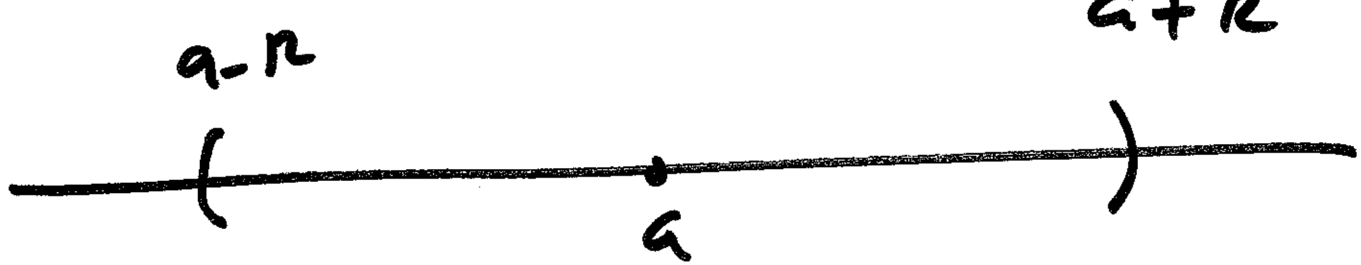
R is the radius of convergence



The convergence at $a+R$ and $a-R$ have to be tested separately.

$$\sum_{n=1}^{\infty} c_n (x-a)^n$$

X does not happen



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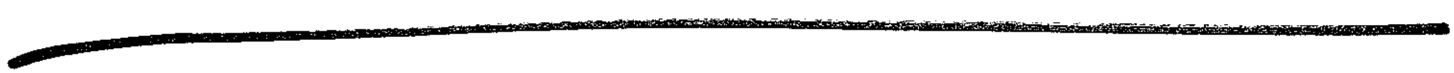

or