Lesson 27   Section 11.9

Representation of Functions by Power Series.

Starting Point: Formula for the Sum of a geometric Series

$$\boxed{1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{provided } |x| < 1}$$

We will use this example to construct several others of functions that are represented by power series.

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\boxed{\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \, , \quad |x| < 1}$$

$$\frac{1}{1+x^2} = \frac{1}{1-\underbrace{(-x^2)}}$$

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1}$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

if $|x|^2 < 1$

or $|x| < 1$.

$$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2}\,\frac{1}{1-x/2}$$

$$\frac{1}{2-x} = \frac{1}{2}\,\frac{1}{1-x/2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(x/2\right)^n \quad \text{if } \left|\frac{x}{2}\right|<1$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n} \quad \text{if } \frac{|x|}{2}<1$$

$$\boxed{\frac{1}{2-x} = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}, \quad |x|<2.}$$

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} =$$

$$= \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{1+x} - \frac{1}{2+x}$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$\frac{1}{2+x} = \frac{1}{2} \frac{1}{1+x/2} = \frac{1}{2} \frac{1}{1-(-x/2)} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}}, \quad \frac{|x|}{2} < 1$$

$$|x| < 2.$$

$$\frac{1}{1+x} - \frac{1}{2+x} = \sum_{n=0}^{\infty} (-1)^n x^n - \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}} \quad \text{for } |x| < 1.$$

$$\frac{1}{x^2+3x+2} = \frac{1}{1+x} - \frac{1}{2+x} = \sum_{n=0}^{\infty} \left[(-1)^n - \frac{(-1)^n}{2^{n+1}} \right] x^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(1 - \frac{1}{2^{n+1}} \right) x^n$$

provided $|x| < 1$.

$$\boxed{\frac{1}{x^2+3x+2} = \sum_{n=0}^{\infty} (-1)^n \left(1 - \frac{1}{2^{n+1}}\right) x^n, \text{ for } |x| < 1.}$$

Theorem: If a power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

converges on an interval $(a-R, a+R)$

(R = radius of convergence)

$$\boxed{\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n (x-a)^n \\ f'(x) &= \sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1} \end{aligned}}$$

converges on $(a-R, a+R)$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

<u>Example:</u> $\sum_{n=0}^{\infty} x^n$ converges for $|x|<1$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\, x^{n-1}.$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

$$\int \frac{1}{1-x}\,dx = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + C$$

$$-\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + c$$

Set $x=0$ $\quad -\ln 1 = C = 0.$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$\boxed{\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}} \qquad |x| < 1$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Switch x to $-x$

$$\ln(1+x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$$

$$\ln(1+x) = -\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$$

$$\boxed{\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad |x|<1}$$

Use this with $\underline{\underline{x}} = -1/2$.

$$\ln(1+x) = \ln(1 - 1/2) = \ln(1/2)$$

$$= -\ln 2 = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(-1/2)^n}{n}$$

$$-\ln 2 = \sum_{n=1}^{\infty} \underbrace{(-1)^{n+1} \cdot (-1)^n}_{\substack{(-1) \cdot (-1)^n \cdot (-1)^n \\ (-1)^{2n}}} \frac{1}{2^n n}$$

$$-\ln 2 = -\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$$

$$\boxed{\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n\,2^n}}$$

$$\ln 2 = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4\,2^4} + \cdots$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad |x| < 1$$

$$\int \frac{dx}{1+x^2} = C + \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\, dx$$

$$\arctan x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad |x| < 1$$

Set $x = 0$; $\arctan \cancel{0} = C + 0$

So $C = 0$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } \underline{|x|<1}.$$

$$x = \frac{1}{\sqrt{3}} \qquad \tan \pi/8 = \frac{1/2}{\sqrt{3}/2} = 1/\sqrt{3}$$

$$\tan \pi/6 = \frac{1}{\sqrt{3}}$$

$$\pi/6 = \arctan \frac{1}{\sqrt{3}} \qquad \boxed{1/\sqrt{3} < 1}$$

$$\pi/6 = \arctan \frac{1}{\sqrt{3}} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \left(\frac{1}{\sqrt{3}}\right)^{2n+1}.$$

$$\boxed{\frac{\pi}{6} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{3}\,3^n(2n+1)}}$$