<u>Lesson 29</u> <u>Section 11.10</u>

<u>Exam 3</u>: April 11, 6:30 PM.
Elliott

Lessons 22 to 30.

The class on Monday <u>4/10</u> - Review.

Taylor and Maclaurin Series

If

$$\underline{\underline{f}}(x) = \sum_{n=0}^{\infty} \underline{\underline{c}}_n (x-a)^n$$

on $|x-a| < R$

then $$c_n = \frac{f^{(n)}(a)}{n!}$$

This power series gets a name = Taylor Series

when $a = 0$, Maclaurin Series.

(Find the power series representation)

Find the Taylor Series of the following function.

(1) $f(x) = e^x + 3e^{2x}$, Maclaurin Series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

<u>Converges for every x</u>.

Evaluate the <u>indefinite</u> integral as an infinite series.

$$\int \frac{\sin x - x}{x^3}\, dx.$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

$$\frac{\sin x - x}{x^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n-2}$$

$$\int \frac{\sin x - x}{x^3}\, dx = \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{2n-2}\, dx \right) + C$$

$$\int \frac{\sin x - x}{x^3}\, dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n-1}}{(2n-1)} + C$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!\,(2n-1)} x^{2n-1} + C.$$

$$e^{3x} = 1 + \frac{3x}{1} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n.$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n.$$

$$e^x + 3e^{2x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + 3 \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$$

$$e^{x} + 3e^{2x} = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!} + 3\,\frac{2^n}{n!}\,x^n \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(1 + 3\, 2^n\right) x^n .$$

Find the Maclaurin Series for

$$f(x) = \frac{\cos 2x - 1}{x^2} .$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

$$= \underline{\underline{1}} - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6$$

$$+ \frac{1}{8!} x^8 + \dots$$

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n} x^{2n}.$$

$$\cos 2x - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n} x^{2n}.$$

$$\frac{\cos 2x - 1}{x^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} 4^n . x^{2n-2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} 4^n x^{2(\underline{\underline{n-1}})}.$$

If one wants the answer as $\cancel{}\ x^{2n}$

$n-1 = k.$, $n = k+1.$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+2)!} 4^{k+1} \cdot x^{2k}.$$

$$\frac{\cos 2x - 1}{x^2} = \frac{-1}{2!} 4 + \frac{1}{4!} 4^2 x^2$$

$$+ \frac{-1}{6!} 4^3 x^4 + \dots$$

$$\lim_{x \to 0} \frac{\cos 2x - 1}{x^2} = -\frac{4}{2} = -2.$$

Use series to approximate

$$\int_0^1 \sin x^4 \, dx$$

with an error $\leq 10^{-3}$.

<u>Solution:</u> $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$$\sin x^4 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+4}}{(2n+1)!}$$

$$\int_0^1 \sin x^4 \, dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{8n+4} \right) dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 x^{8n+4} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{(8n+5)} x^{8n+5} \Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{(8n+5)}.$$

$$S = \sum_{n=1}^{\infty} (-1)^n b_n$$

$$|S - S_N| \leq \underline{\underline{b_{N+1}}} \leq 10^{-3}$$

$$S_N = \sum_{n=1}^{N} (-1)^n b_n$$

$$b_n = \frac{1}{(8n+5)(2n+1)!}$$

$$b_{N+1} = \frac{1}{(8N+13)(2N+3)!} \leq \frac{1}{10^3}$$

$$(8N+13)\,(2N+3)! \geq 10^3.$$

$\underline{N=1}$: $21 \cdot 5! = 21 \cdot 120 > 10^3.$