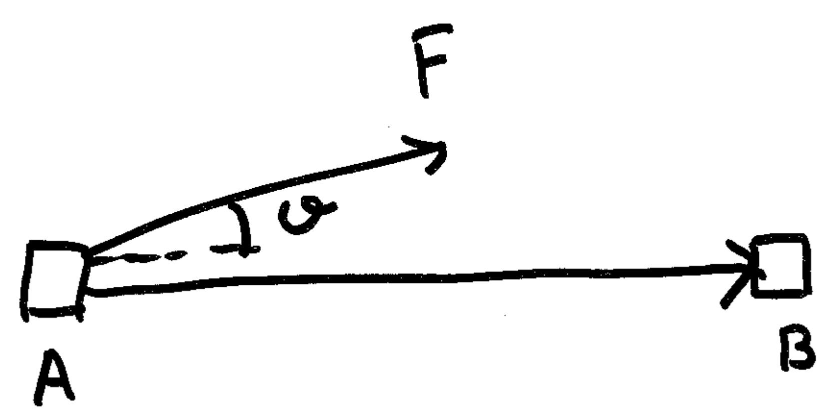
Lesson 3 Section 12.3

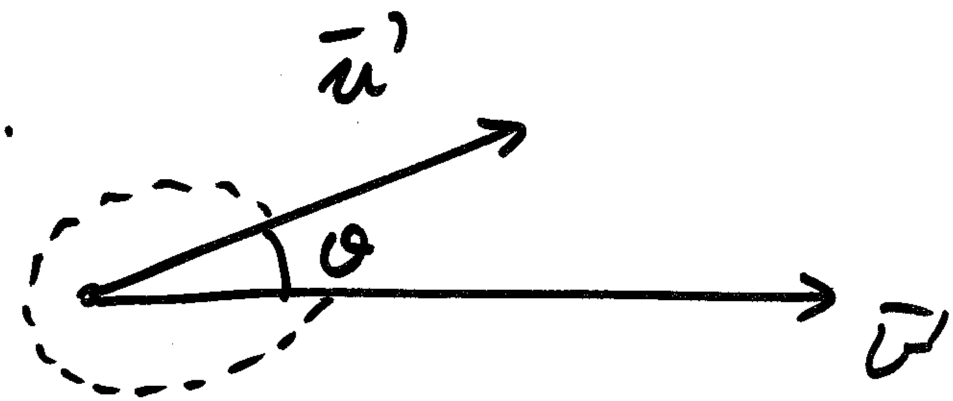
The Dot Product.

Physical Motivation



Work: $|\vec{F}| \cdot |\overrightarrow{AB}| \cdot \cos\theta$.

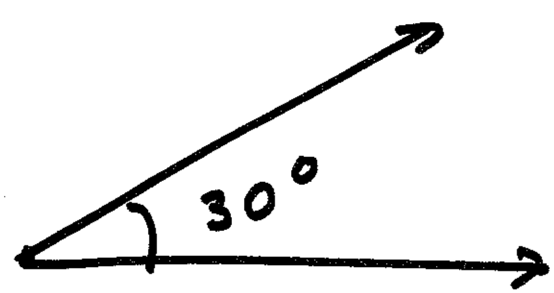
The Dot product of $\vec{u}$ and $\vec{v}$



$$\vec{u} \cdot \vec{v} = |\vec{u}| . |\vec{v}| . \cos\theta$$

θ = angle between $\vec{u}$ and $\vec{v}$, $\theta \leq 180^\circ$
or π radians

$\vec{v}$, $|\vec{v}| = 5$

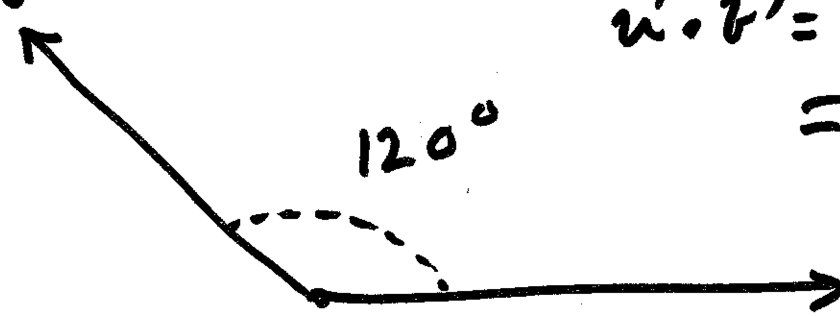


$\vec{u}$, $|\vec{u}| = 10$

$\vec{u} \cdot \vec{v} = 5 \cdot 10 \cdot \cos 30^\circ = 50 \cdot \frac{\sqrt{3}}{2}$.

$\vec{v}$

$|\vec{v}| = 5$



$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos 120$

$= -5 \cdot 10 \cdot \frac{1}{2} = -25.$

$\vec{u}$, $|\vec{u}| = 10$

If $\vec{u} = \langle u_1, u_2, u_3 \rangle = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$

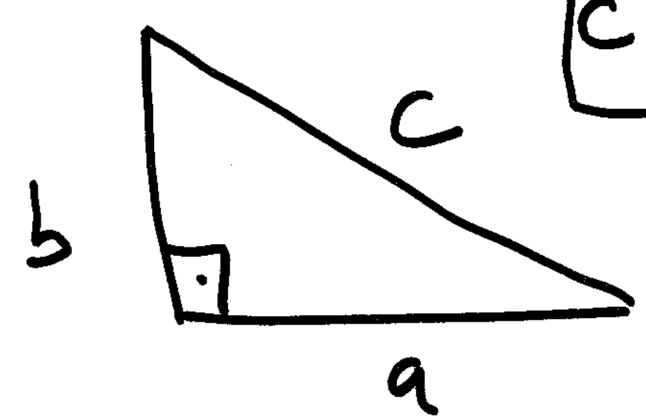
$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$.

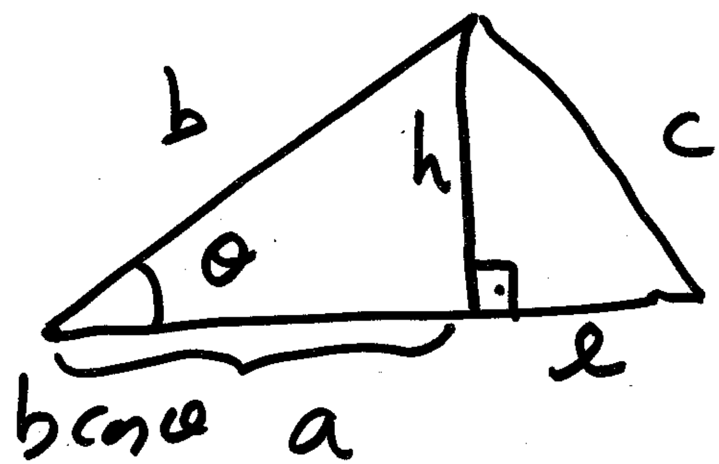
How do I compute $\vec{u} \cdot \vec{v}$?

$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos \theta$

<u>High School Geometry</u>:

$$\boxed{c^2 = a^2 + b^2}$$





$$c^2 = h^2 + \ell^2$$

$$h = b\sin\theta$$

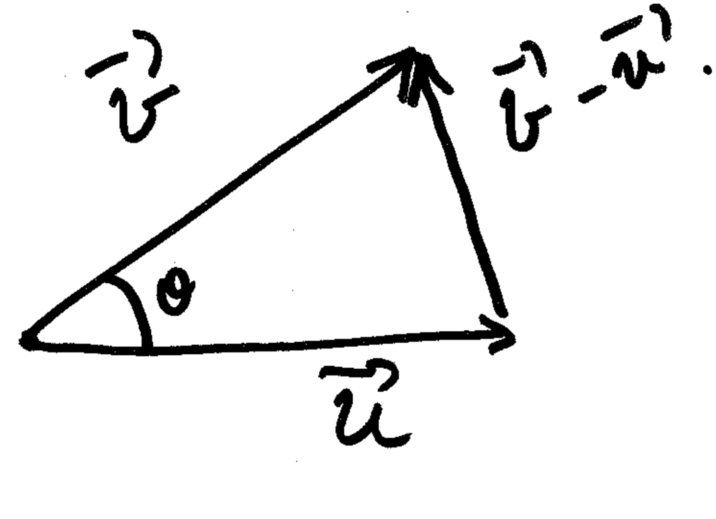
$$\ell = a - b\cos\theta$$

$$\begin{aligned} c^2 &= b^2\sin^2\theta + (a - b\cos\theta)^2 \\ &= \underbrace{b^2\sin^2\theta}_{} + a^2 + \underbrace{b^2\cos^2\theta}_{} - 2ab\cos\theta. \end{aligned}$$

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta}$$

when $\theta = \pi/2$, $c^2 = a^2 + b^2$.

How do I use this when dealing with vectors?



$a = |\vec{u}|$

$b = |\vec{v}|$

$c = |\vec{v} - \vec{u}|$

$$\boxed{|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\underbrace{|\vec{u}|.|\vec{v}|.\cos\alpha}_{} \\ -2\,\vec{u}\cdot\vec{v}}$$

$$\vec{u} = \langle u_1, u_2, u_3 \rangle, \quad \vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$\vec{u}-\vec{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle.$$

$$\underbrace{(u_1-v_1)^2}_{} + (u_2-v_2)^2 + (u_3-v_3)^2 = u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - 2\,\vec{u}\cdot\vec{v}$$

$$\boxed{\vec{u}\cdot\vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3}$$

$$(u_1 - v_1)^2 = u_1^2 + v_1^2 - \underline{\underline{2u_1}} v_1$$

$$-2u_1v_1 - 2u_2v_2 - 2u_3v_3 = -2\, u \cdot v$$

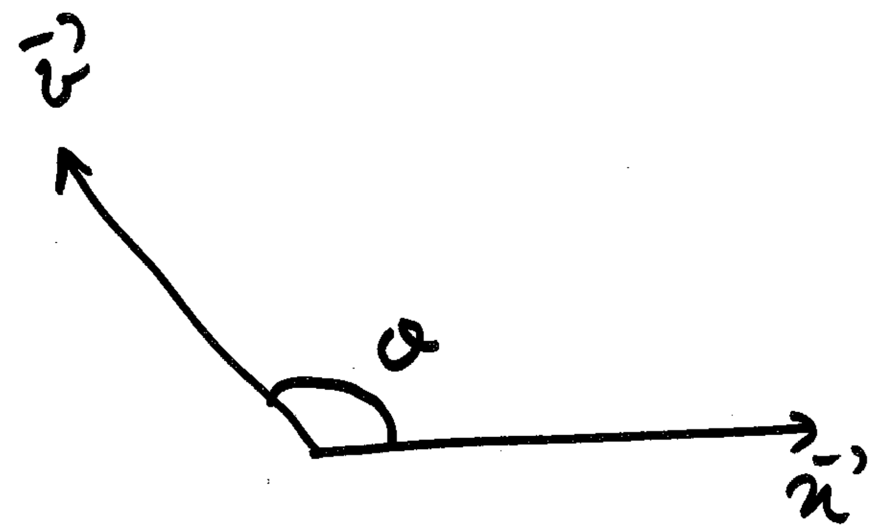
<u>Example</u>: Let $\vec{u} = \langle 1, 2, 5 \rangle$

$\vec{v} = \langle -1, 3, -8 \rangle$

Find $\vec{u} \cdot \vec{v}$.

$$\vec{u} \cdot \vec{v} = 1 \cdot (-1) + 2 \cdot 3 + 5 \cdot (-8) = -35.$$

Find the angle between $\vec{u}$ and $\vec{v}$



$$\vec{u} \cdot \vec{v} = |\vec{u}| . |\vec{v}| . \cos\alpha$$

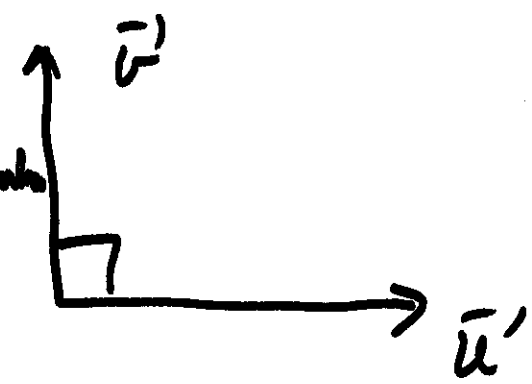
$$-35 = \sqrt{1+4+25} . \sqrt{1+9+64} . \cos\alpha$$

$$-35 = \sqrt{30} . \sqrt{74} . \cos\alpha$$

$$\boxed{\cos\alpha = \frac{-35}{\sqrt{30} . \sqrt{74}}}$$

Projections.

Remark: If $\vec{u}$ and $\vec{v}$ are perpendicular



$$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta = 0$$

Ex: $\vec{u} = \langle 1, -1, 2 \rangle$

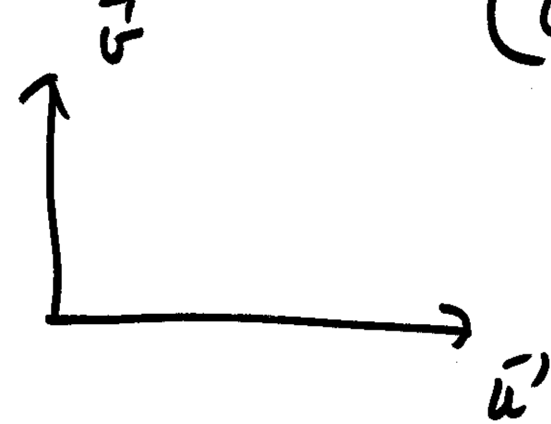
$\vec{v} = \langle 2, 2, 1 \rangle$

$\vec{u} \cdot \vec{v} = 2 - 2 + 2 = 2$. Not perpendicular

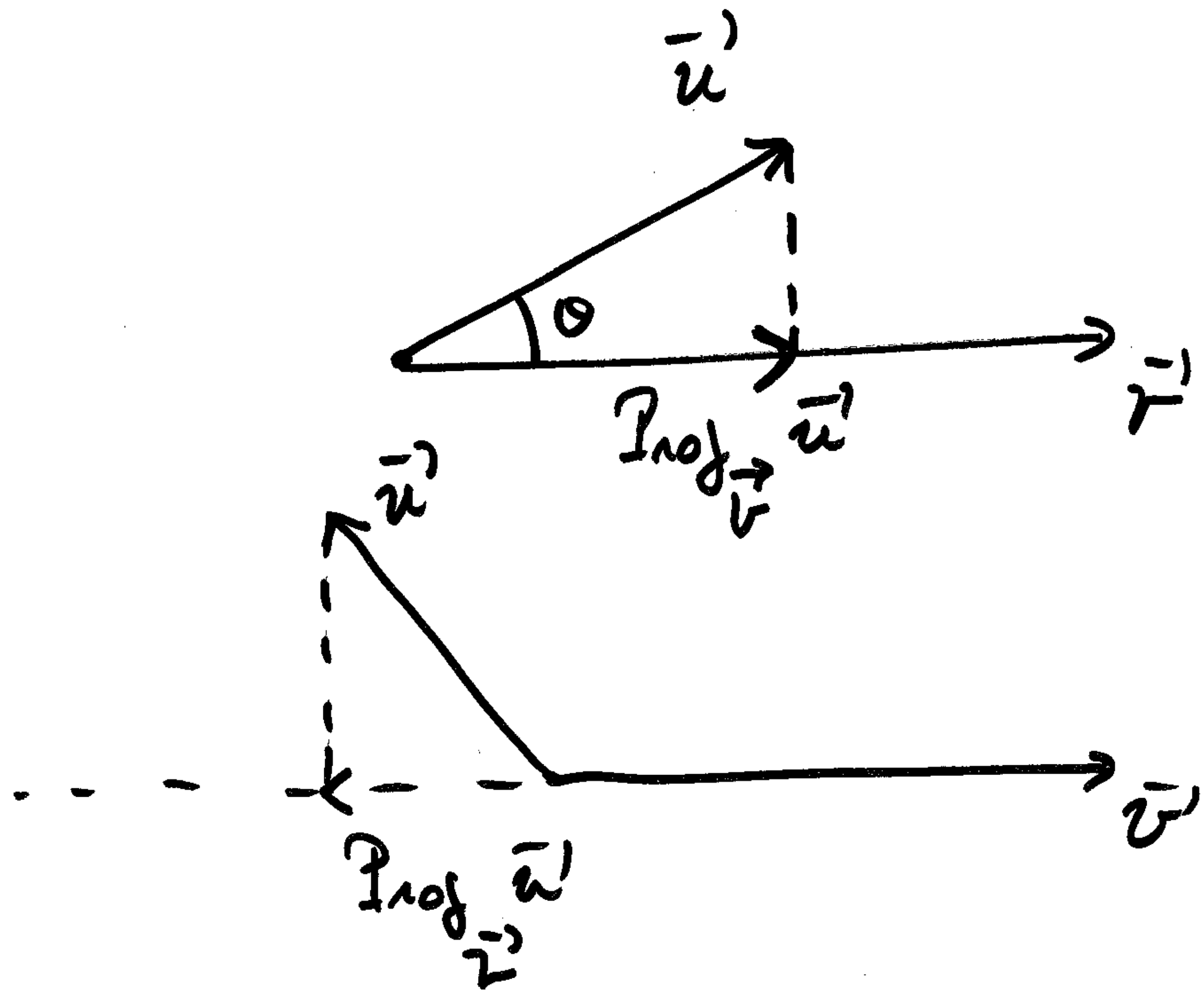
$\vec{u} = \langle 1, 1, 2 \rangle$

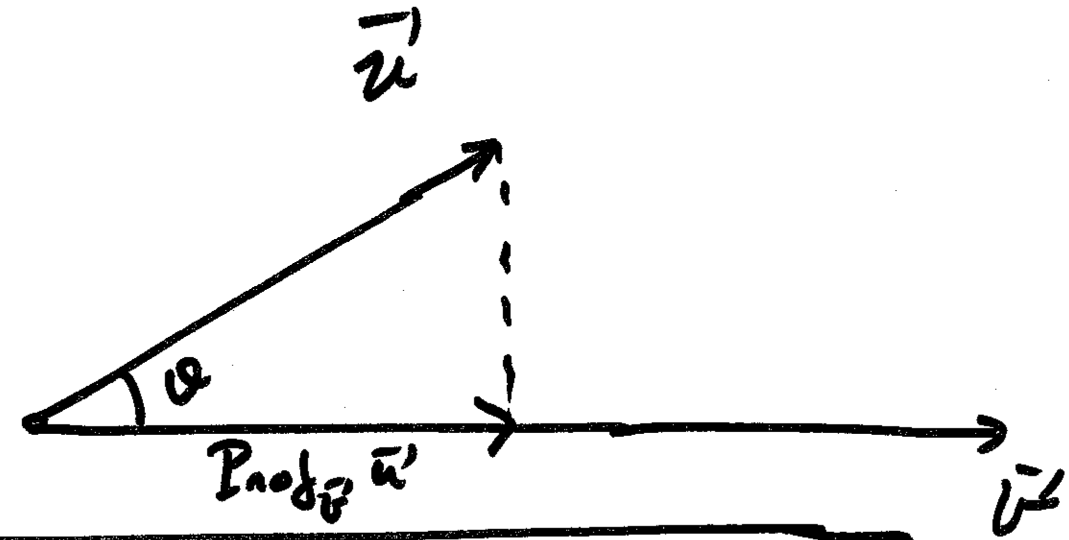
$\vec{v} = \langle -2, 0, 1 \rangle$

$\vec{u} \cdot \vec{v} = -2 + 0 + 2 = 0$ Are perpendicular (orthogonal)



Projections :





$$\boxed{\text{Proj}_{\vec{v}}\,\vec{u} = \frac{|\vec{u}|\cdot\cos\theta\,|\vec{v}|}{|\vec{v}|}\,\frac{\vec{v}}{|\vec{v}|}.}$$

Compute this in coordinates or in terms of the dot product.

$$\mathrm{Proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\cdot\frac{\vec{v}}{|\vec{v}|}.$$

$$\boxed{\mathrm{Proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\cdot\vec{v}}$$

<u>Example:</u> Let $\vec{u} = \langle -1, 3, -5 \rangle$

$\vec{v} = \langle 0, 1, 2 \rangle$

Compute $\mathrm{Proj}_{\vec{v}}\,\vec{u}$. and $\mathrm{Proj}_{\vec{u}}\,\vec{v}$.

$$\text{Proj}_{\vec{v}}\,\vec{u} = \frac{-7}{5}\cdot\langle 0,1,2\rangle$$

$$\text{Proj}_{\vec{u}}\,\vec{v} = \frac{-7}{35}\cdot\langle -1,3,-5\rangle.$$

$$\left\{\begin{aligned} &\vec{u}\cdot\vec{v} = -7 \\ &|\vec{u}| = \sqrt{1+9+25} \\ &\quad\;\; = \sqrt{35} \\ &|\vec{v}| = \sqrt{5}. \end{aligned}\right.$$

$$\vec{u}\cdot\vec{v} = -7 = |u|\cdot|v|\cdot\cos\alpha$$