Lesson 30   Section 11.10

Exam 3: Lessons 22 to 30

The Binomial Series

Express $(1+x)^k$ as a power series.

k is a real number. Examples:

$(1+x)^{1/2}$, $(1+x)^{1/4}$, . $(1+x)^{-1}$, ..

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MY MA162 CLASS NOTES.

$\underline{k=-1}$

$$\frac{1}{1+x} = (1+x)^{-1} = \underbrace{\sum_{n=0}^{\infty} (-1)^n x^n}.$$

$\underline{k=-3}$: $(1+x)^{-3}$

$$\frac{d}{dx}\frac{1}{1+x} = \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2}$$

$$\frac{1}{(1+x)^2} = -\frac{d}{dx}\frac{1}{1+x}$$

$$\frac{d^2}{dx^2}\left(\frac{1}{1+x}\right) = 2(1+x)^{-3}$$

$$(1+x)^{-3} = \frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{1+x}\right)$$

$$\frac{1}{(1+x)^3} = \frac{1}{2}\frac{d^2}{dx^2}\frac{1}{1+x} =$$

$$= \frac{1}{2}\frac{d^2}{dx^2}\sum_{n=0}^{\infty}(-1)^n x^n$$

$$\frac{d^2}{dx^2} x^n = n(n-1)\, x^{n-2}$$

$$\frac{1}{(1+x)^3} = \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n\, n(n-1)\, x^{\boxed{n-2}}.$$

Set $k = n-2$; $n = k+2$

$$(-1)^{k+2} = (-1)^k$$

$$(1+x)^{-3} = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k (k+2)(k+1)\, x^k.$$

$$\boxed{(1+x)^{-3} = \sum_{k=0}^{\infty} (-1)^k \frac{(k+2)(k+1)}{2}\, x^k.}$$

Same idea works for $(1+x)^{-m}$, $m = 1, 2, 3, \ldots$

$(1+x)^{1/2}$ or $(1+x)^{1/3}$, etc.

Find the Maclaurin Series of

$f(x) = (1+x)^k$, k a real number.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f(x) = (1+x)^k$, <u>find $f^{(n)}(0)$</u>.

$f(0) = 1$. ; $f'(x) = k(1+x)^{k-1}$, $f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$; $f''(0) = k(k-1)$.

$$f^{(3)}(x) = k(k-1)(k-2)(1+x)^{k-3}$$

$$f^{(3)}(0) = k(k-1)(k-2)$$

$$f^{(4)} = k(k-1)(k-2)(k-3)$$

In general:

$$f^{(n)}(0) = k(k-1)(k-2)\dots(k-(n-1))$$

$$\boxed{f^{(n)}(0) = k(k-1)(k-2)\dots(k-n+1)}$$

$f(x) = (1+x)^k$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2$$

$$+ \frac{k(k-1)(k-2)}{3!} x^3 + \dots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n$$

$$+ \dots$$

One can prove that if $|x| < 1$

The binomial Series

$$\underline{\underline{(1+x)^k}} = 1 + kx + \frac{k(k-1)}{2!} x^2 + \dots$$

$$+ \frac{k(k-1)(k-2)\dots(k-n+1)}{n!} x^n + \dots$$

Examples:

(1) Find the binomial series of

$(1+x)^{1/2}$. $\qquad k = 1/2$.

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2$$

$$+ \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \cdots$$

$$+ \boxed{\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!}} x^n + \cdots$$

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\cdots\overset{n-1}{\left(\frac{1}{2}-n+1\right)}}{n!}$$

$$= \frac{\frac{1}{2}\left(\frac{1-2}{2}\right)\left(\frac{1-4}{2}\right)\left(\frac{1-6}{2}\right)\cdots\left(\frac{1-2n+2}{2}\right)}{n!}$$

$$= \frac{\frac{1}{2^n}(-1)(-3)(-5)\cdots(3-2n)}{n!}$$

$$= \frac{(-1)^{n-1}}{2^n n!}\,1.3.5.7\ldots(2n-3).$$

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{2^n n!} 1.3.5 \ldots (2n-3) x^n$$

Find the Maclaurin series of

$$f(x) = \frac{x^4}{\sqrt{5+x}} = \frac{x^4}{\sqrt{5}} \boxed{\frac{1}{(1+\frac{x}{5})^{1/2}}}$$

<u>Begin with:</u> $(1+x)^{-1/2}$

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!} x^2$$

$$+ \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!} x^3$$

$$+ \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} x^n.$$

<u>Examine:</u> $\dfrac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}$

$$= \frac{-\frac{1}{2}\left(\frac{-1-2}{2}\right)\left(\frac{-1-4}{2}\right)\left(\frac{-1-6}{2}\right)\cdots\left(\frac{-1-2n+2}{2}\right)}{n!}$$

$$= \frac{(-1)(-3)(-5)(-7)\cdots(1-2n)}{2^n\, n!}.$$

$$= \frac{(-1)^n}{2^n\, n!}\, 1.3.5.7\cdots(2n-1).$$

<u>Conclusion:</u>

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \sum_{n=2}^{\infty} \frac{(-1)^n}{2^n\, n!}\, 1.3.5.7\cdots(2n-1)x^n$$

$$\left(1+\frac{x}{5}\right)^{-1/2} = 1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n\, n!}\, \frac{1.3.5.7\cdots(2n-1)}{5^n}.\; x^n$$

Find the Maclaurin Series of

$$\sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt.$$

<u>Begin with:</u> $(1+t)^{-1/2}$

(ii) Use this to find the power series of

$$(1-t^2)^{-1/2}$$

(iii) Integrate the power series to find $\sin^{-1} x$.