Lesson 32 Section 10.2

Calculus with parametric curves.

Goal: Given a curve

$$x = x(t), \quad y = y(t)$$

(1) Find an equation of the line tangent to the curve at a point on the curve

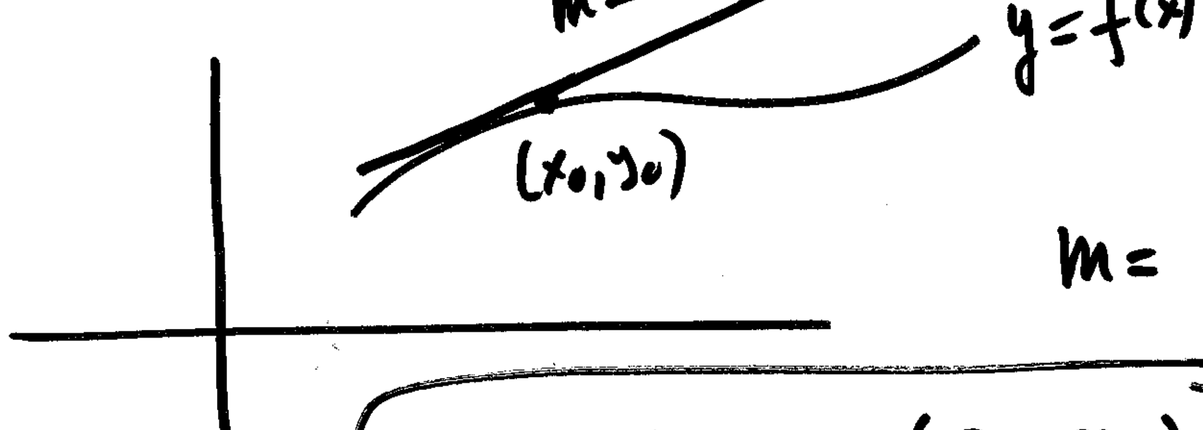
2) Analyze the concavity of the curve.

3) Compute its length.

<u>Basic example:</u> $x = t, \quad y = f(t)$

or $y = f(x)$

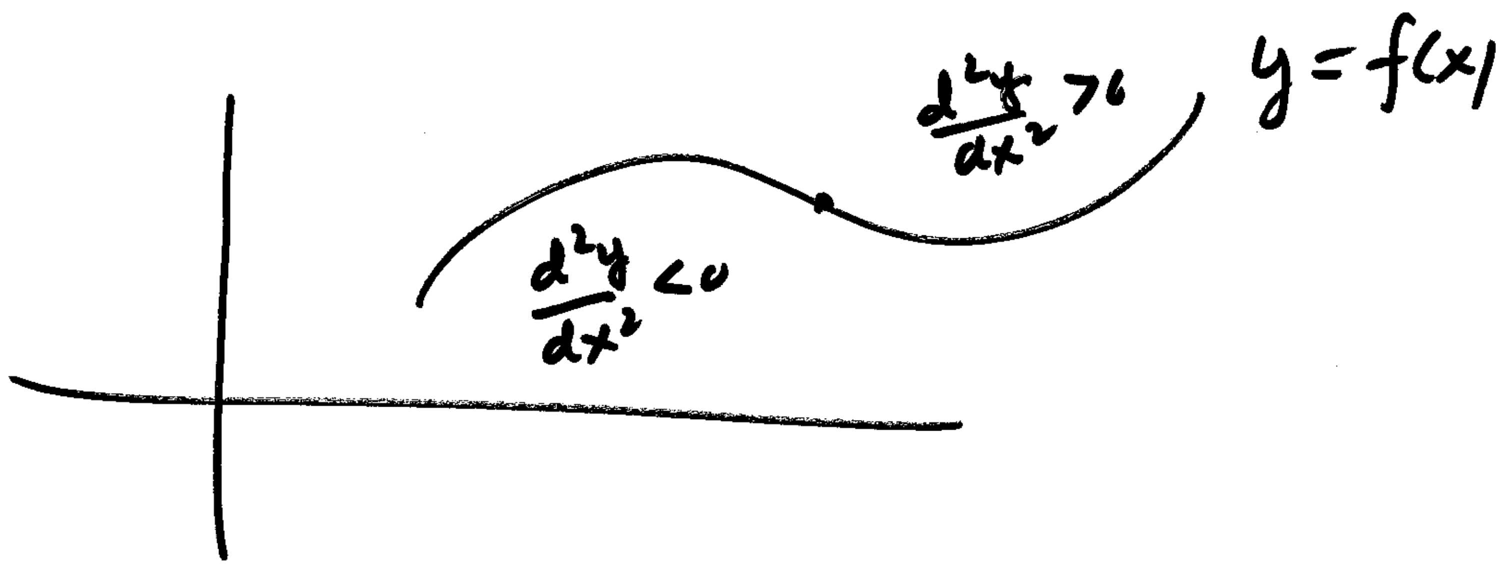
$m =$ slope of the tangent line



$$m = \frac{df}{dx}(x_0)$$

$$\boxed{y - y_0 = m(x - x_0)}$$

To analyze the concavity we study the sign of $\frac{d^2y}{dx^2}$.



Now we have a curve

$y = y(t)$ and $x = x(t)$

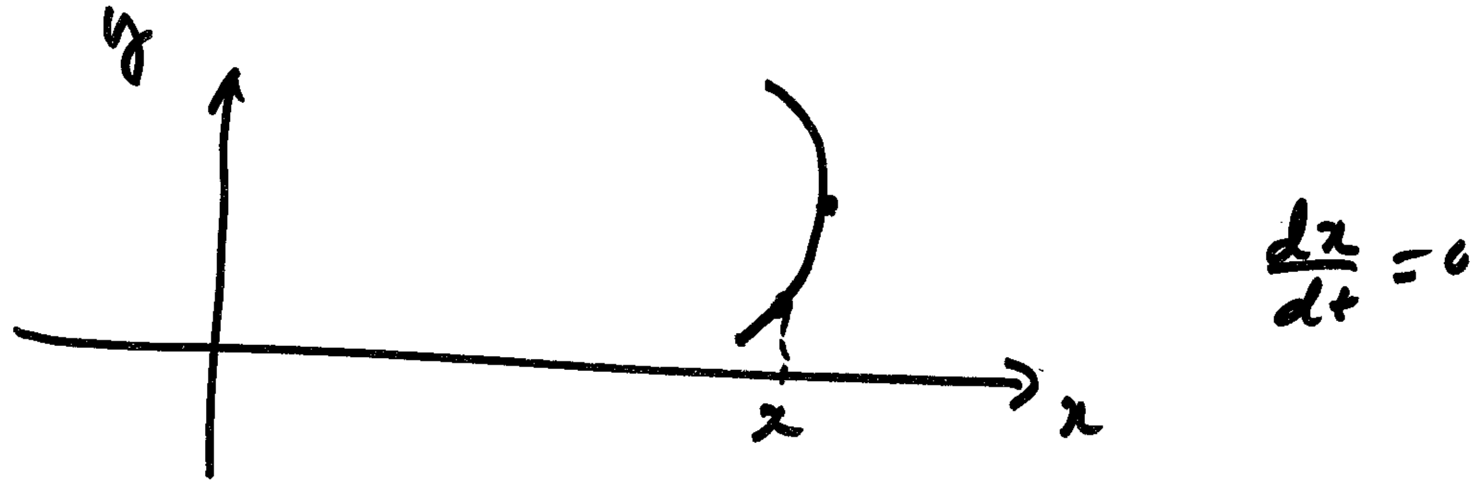
and we want to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$x = x(t)$ and $y = y(t)$

we want to find $\frac{dy}{dx}$.

$x = x(t)$ can we write $t = t(x)$?

Yes as long as $\frac{dx}{dt} \neq 0$.



$\frac{dx}{dt} = 0$, tangent is vertical.

We will assume that $\frac{dx}{dt} \neq 0$

In this case we can write
$t = t(x)$

$$y = y(t) = y(t(x))$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}}$$

But $\dfrac{dt}{dx} = \dfrac{1}{\dfrac{dx}{dt}}$

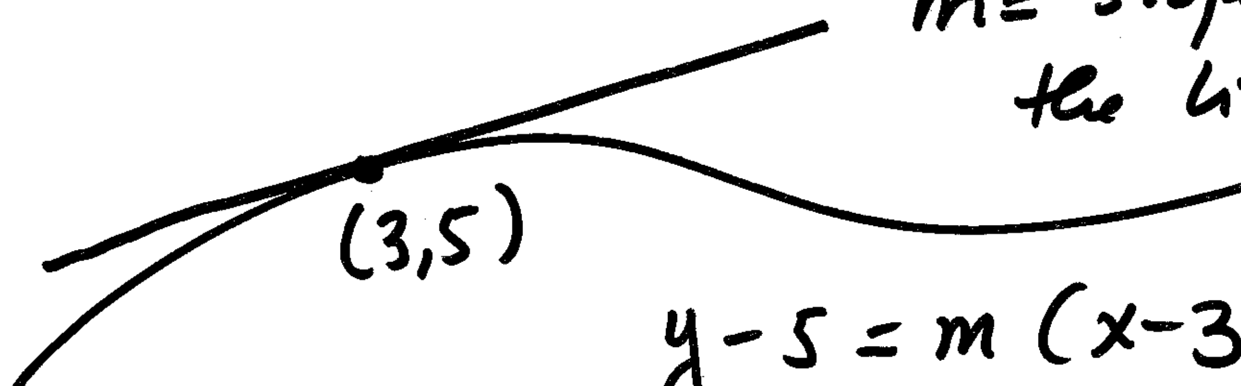
$$\boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}}$$

<u>Example:</u> Find an equation for the line tangent to the curve

$$x(t) = t^2 + t + 1$$

$$y(t) = t^3 + 4t$$

at the point $(3, 5)$



$m =$ slope of the line.

$$y - 5 = m(x - 3)$$

$$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{at } (3,5)$$

$$\boxed{m = \frac{3t^2+4}{2t+1}}, \quad 2t+1 \neq 0$$

$$m = \frac{7}{3}, \quad t=1$$

We have to figure out the value of t.

$$x = t^2 + t + 1 = 3$$

$$y = t^3 + 4t = 5$$

$$\boxed{t=1}$$

$$t^2 + t - 2 = 0$$

$$(t-1)(t+2) = 0$$

$$\boxed{t=1, \quad t=-2.}$$

How do we find $\frac{d^2y}{dx^2}$?

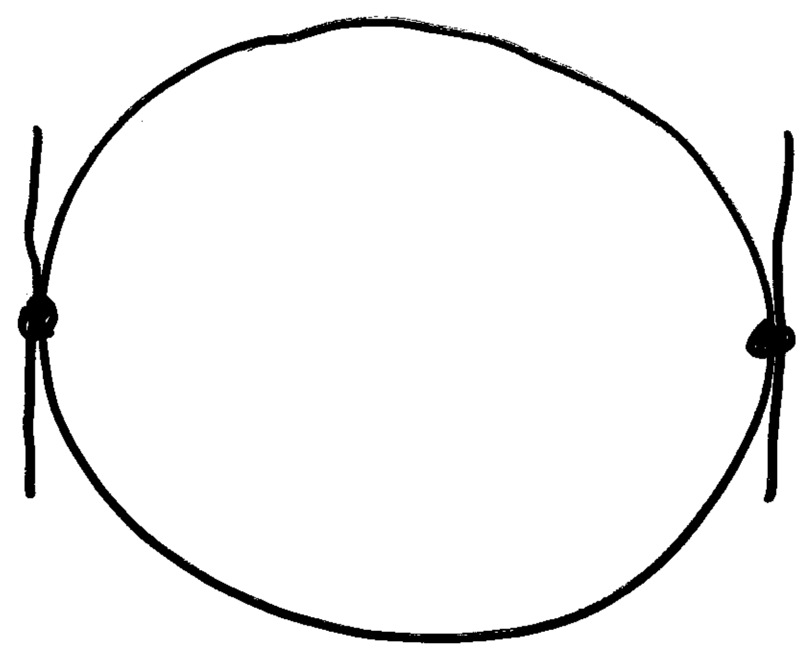
$$\frac{d\overset{\frac{dy}{dx}}{\overset{\downarrow}{y}}}{dx} = \frac{\frac{d\overset{\frac{dy}{dx}}{\overset{\downarrow}{y}}}{dt}}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0$$

Apply this to $\frac{dy}{dx}$ instead of y

$$\boxed{\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0}$$

Example: $x = \cos t, \quad y = \sin t$

$$0 \le t \le 2\pi$$



$$\frac{dx}{dt} = -\sin t$$

$$\frac{dx}{dt} = 0 = -\sin t$$

$$t = 0, \; t = \pi$$

$$t = 2\pi$$

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

$$\frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dt} = \cos t$$

$$\frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\frac{\cos t}{\sin t}, \quad \underline{\sin t \neq 0}$$

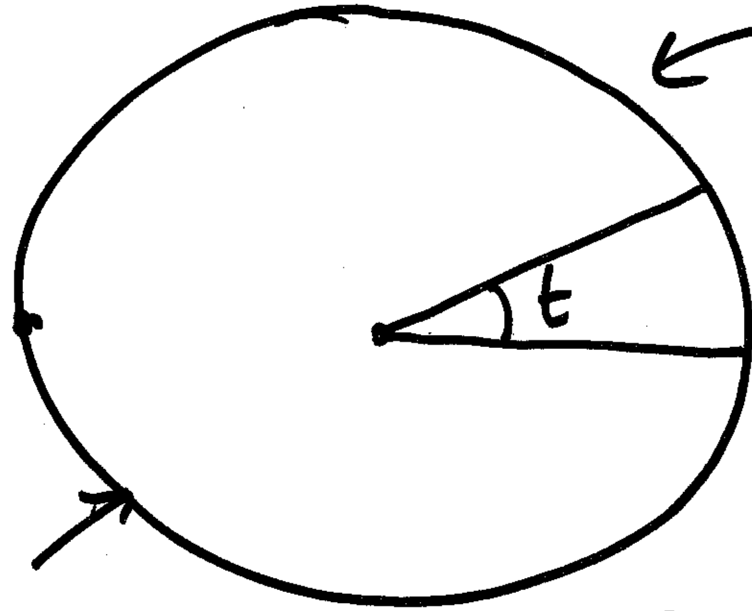
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} =$$

$$\frac{dy}{dx} = -\frac{\cos t}{\sin t}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -\frac{-\sin^2 t - \cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1}{\sin^2 t}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{1}{\sin^2 t}}{-\sin t} = -\frac{1}{\sin^3 t}}$$



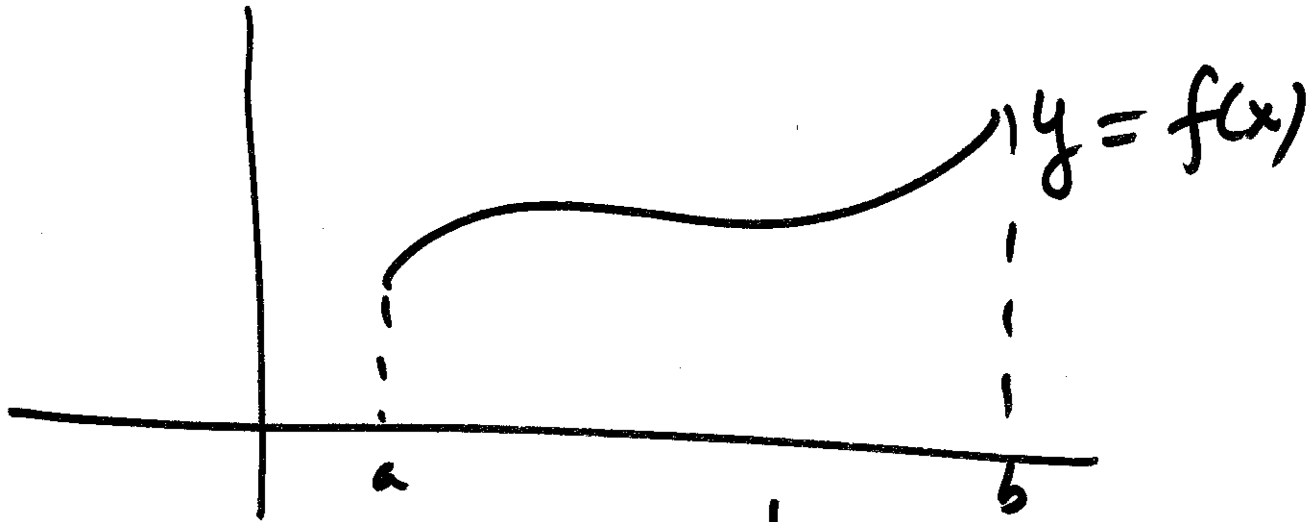
$0 < t < \pi$

$\sin t > 0$

$\frac{d^2y}{dx^2} = -\frac{1}{\sin^3 x} < 0$

$\pi < t < 2\pi$, $\sin t < 0$

$\frac{d^2y}{dx^2} > 0.$



$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

We want to generalize this to the case when the curve is given by $x = x(t)$, $y = y(t)$.

Example: $x = \frac{1}{2} t^2; \quad y = \frac{1}{3} t^3$

$0 \le t \le 1$

$$\frac{dx}{dt} = t; \quad \frac{dy}{dt} = t^2$$

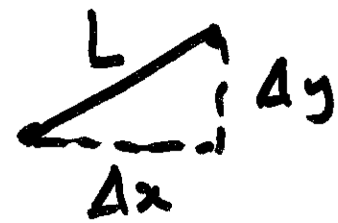
$$L = \int_0^1 \sqrt{t^2 + t^4}\, dt$$

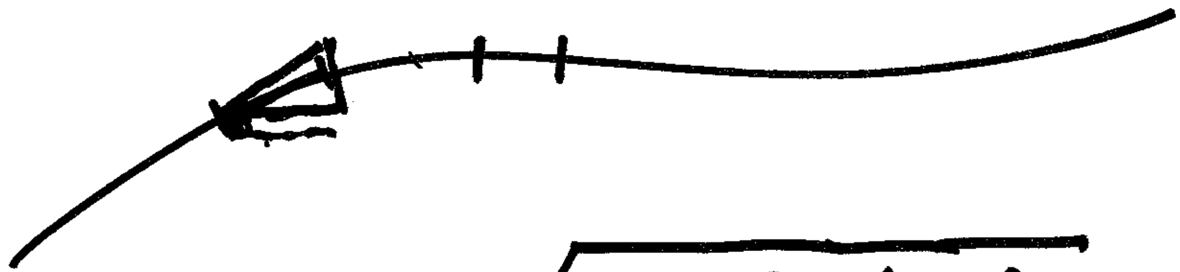
$$= \int_0^1 \underline{\underline{t}} \sqrt{1 + t^2}\, \underline{\underline{dt}}$$

$u = 1 + t^2$
$du = 2t\,dt$

$$= \int_1^2 u^{1/2} \cdot \frac{du}{2} = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_1^2 =$$

$$= \frac{1}{3}(2\sqrt{2} - 1).$$





$$L = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \quad \Delta t.$$

$$\frac{\Delta x}{\Delta t} \sim \frac{dx}{dt}, \quad \frac{dy}{dt} \sim \frac{\Delta y}{\Delta t}$$

$$\boxed{L = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt}$$