Lesson 4. Section 12.4

The Cross Product.

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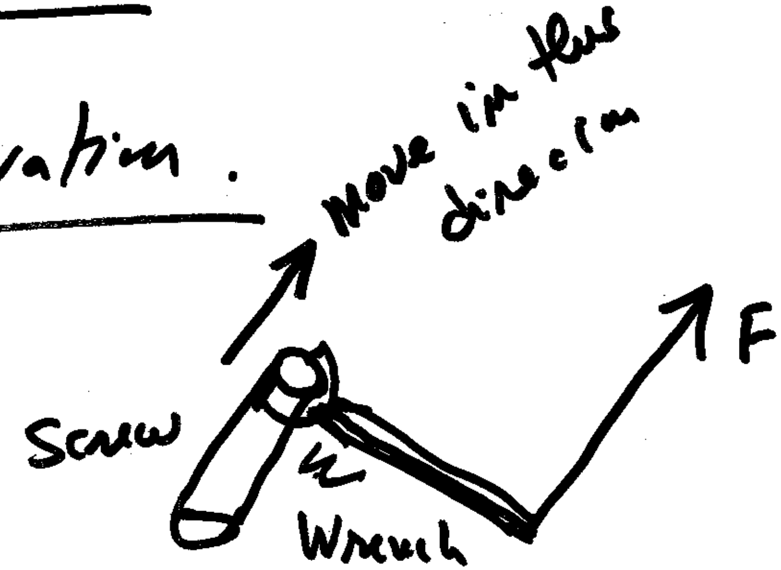
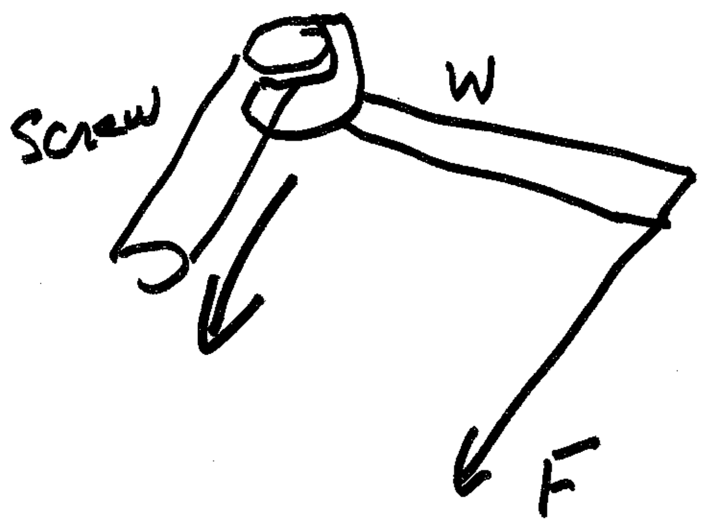
The Cross Product

Physical Motivation.



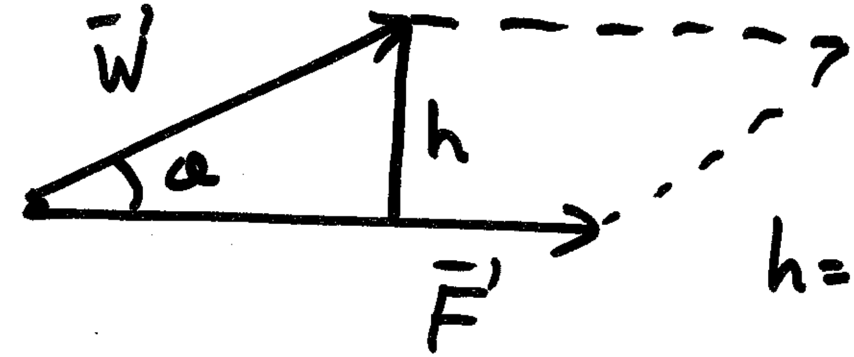


The cross product of $\vec{F}$ and $\vec{W}$, $\vec{F} \times \vec{W}$ is ~~a~~ THE vector with the following properties

(i) $\vec{F} \times \vec{W}$ is perpendicular to both $\vec{F}$ and $\vec{W}$

(ii) The orientation is given by the right hand rule

(iii)



$h = |\vec{w}| . \sin\alpha$

The length of the vector $\vec{F} \times \vec{w}$

= Area of the parallelogram determined by $\vec{F}$ and $\vec{w}$

$= |\vec{F}| . h = |\vec{F}| . |\vec{w}| . \sin\alpha .$

<u>Question:</u> How do we compute $\vec{F} \times \vec{W}$ using Coordinates?

Given $\vec{u} = \langle u_1, u_2, u_3 \rangle = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$

$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$.

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - v_2 u_3 ,\ u_3 v_1 - u_1 v_3 ,\ u_1 v_2 - u_2 v_1 \rangle.$$

Use determinants to remember this formula:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

Ex:

$$\begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix} = 7 - 15 = -8$$

$$\begin{vmatrix} \boxed{a_1} & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example: Compute

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 2 & 1 & 4 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= 1 \cdot (4-3) - 1 \cdot (0-6) + 2 \cdot (0-2)$$

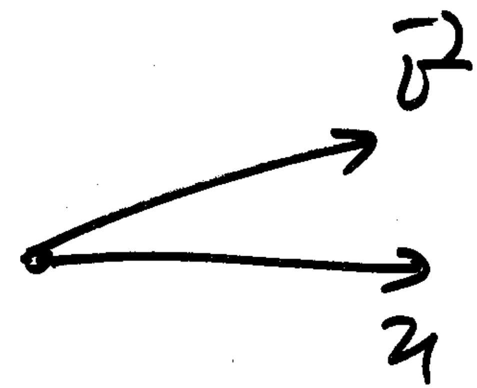
$$= +1 + 6 - 4 = 3$$

$\vec{u} = \langle u_1, u_2, u_3 \rangle$; $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

The order is important!

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$



<u>Example:</u> Compute $\vec{u} \times \vec{v}$ where

$\vec{u} = \langle 1, 2, 1 \rangle$, $\vec{v} = \langle 0, 1, 2 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

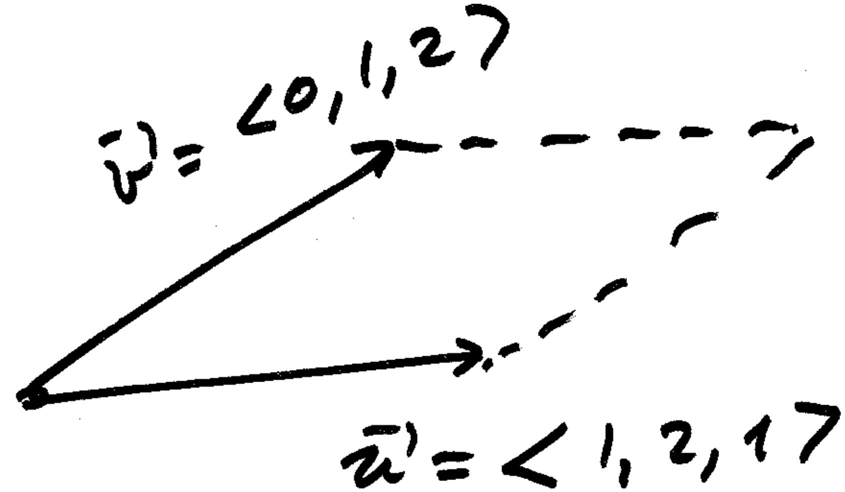
$$= 3\vec{i} - 2\vec{j} + \vec{k}.$$

Verify that indeed $\vec{u} \times \vec{v} = - \vec{v} \times \vec{u}$.

$$\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= -3\vec{i} + 2\vec{j} - \vec{k}.$$

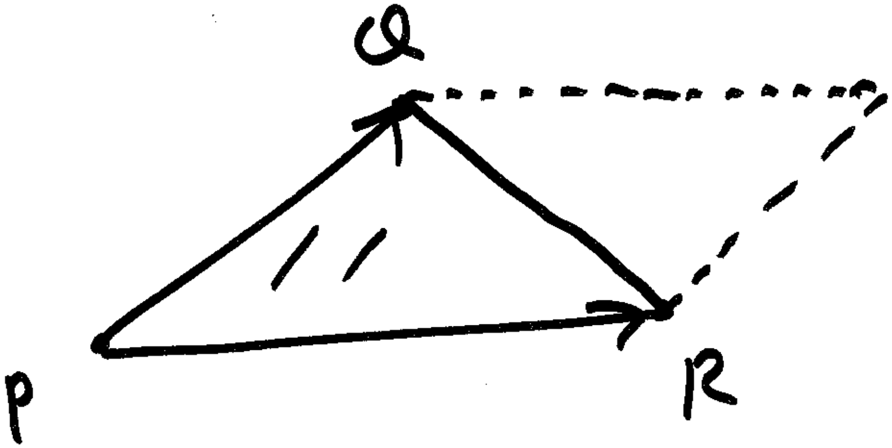


Find the area of the parallelogram defined by $\vec{u}$ and $\vec{v}$.

$|\vec{u} \times \vec{v}| =$ Area of this parallelogram.

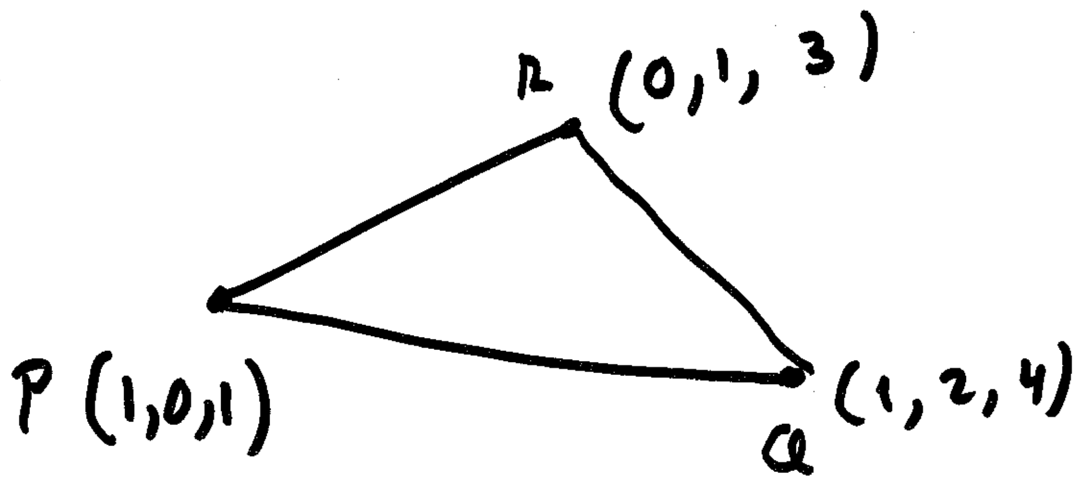
$\vec{u} \times \vec{v} = 3\vec{i} - 2\vec{j} + \vec{k}$

$|\vec{u} \times \vec{v}| = \sqrt{9+4+1} = \sqrt{14}$



Area of the triangle $= \frac{1}{2}$ Area of the ~~parallo~~ parallelogram.

$$= \frac{1}{2} \left| \overline{PR} \times \overline{PQ} \right|.$$



$$\vec{PQ} = \langle 0, 2, 3 \rangle = 2\vec{j} + 3\vec{k}$$

$$\vec{PR} = \langle -1, 1, 2 \rangle = -\vec{i} + \vec{j} + 2\vec{k}.$$

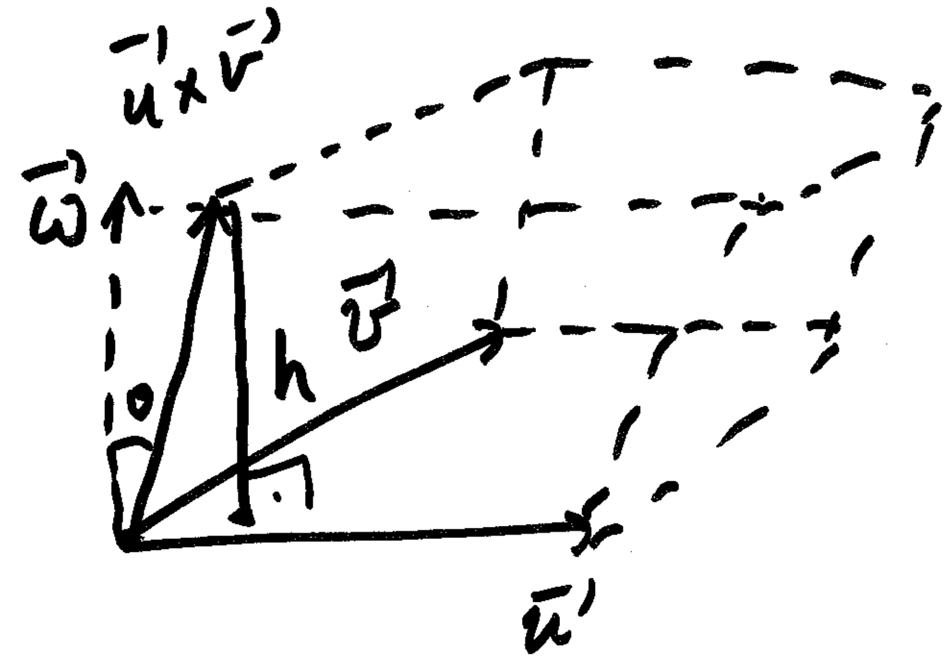
$$A = \frac{1}{2} \left| (2\vec{j} + 3\vec{k}) \times (-\vec{i} + \vec{j} + 2\vec{k}) \right|$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 3 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= \vec{i} - \vec{j}(3) + \vec{k}\,2 = \vec{i} - 3\vec{j} + 2k.$$

$$A = \frac{1}{2}\sqrt{1+9+4} = \frac{\sqrt{14}}{2}.$$

Parallelepiped.



$$\text{Volume} = \underbrace{\text{Area of the base}} \times \text{height}$$

$$= |\vec{u} \times \vec{v}| \cdot |\vec{w}| \cdot \underbrace{\cos\theta}$$

θ = angle between $\vec{w}$ and $\vec{u} \times \vec{v}$.

$$\text{Volume} = \left| (\vec{u} \times \vec{v}) \cdot \vec{w} \right|$$

$$= \left| \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right|$$

Example: Find the ~~area~~ volume of the parallelepiped defined by $\vec{a} = \vec{i} + \vec{j} + 2\vec{k}$

$\vec{b} = -\vec{i} - \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + 3\vec{k}$.

Solution:

$$\text{Volume} = |\vec{a} \cdot (\vec{b} \times \vec{c})| = \left| \begin{vmatrix} 1 & 1 & 2 \\ -1 & -1 & 1 \\ 1 & 0 & 3 \end{vmatrix} \right|$$

$$= 1 \cdot (-3-0) - 1 \cdot (-3-1) + 2(0+1)$$

$$= -3 + 4 + 2 = 3.$$