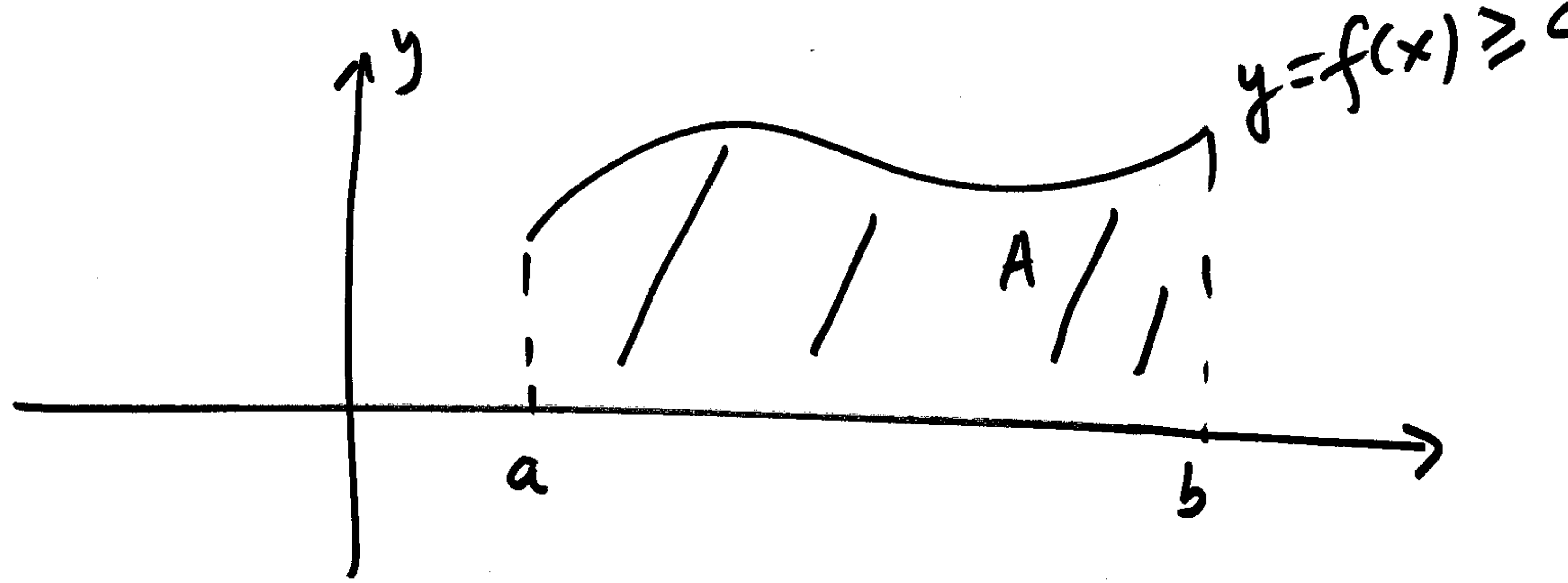
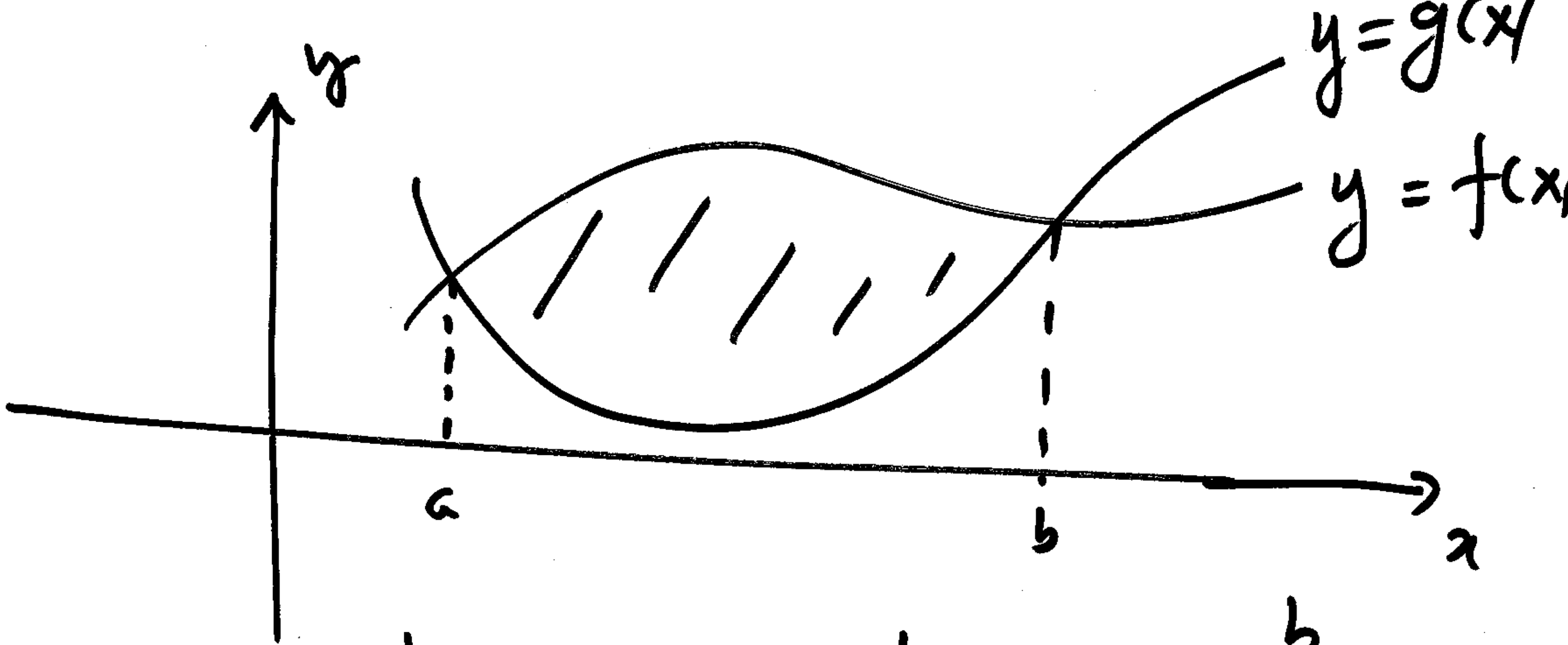
Lesson 5. Areas Between Curves

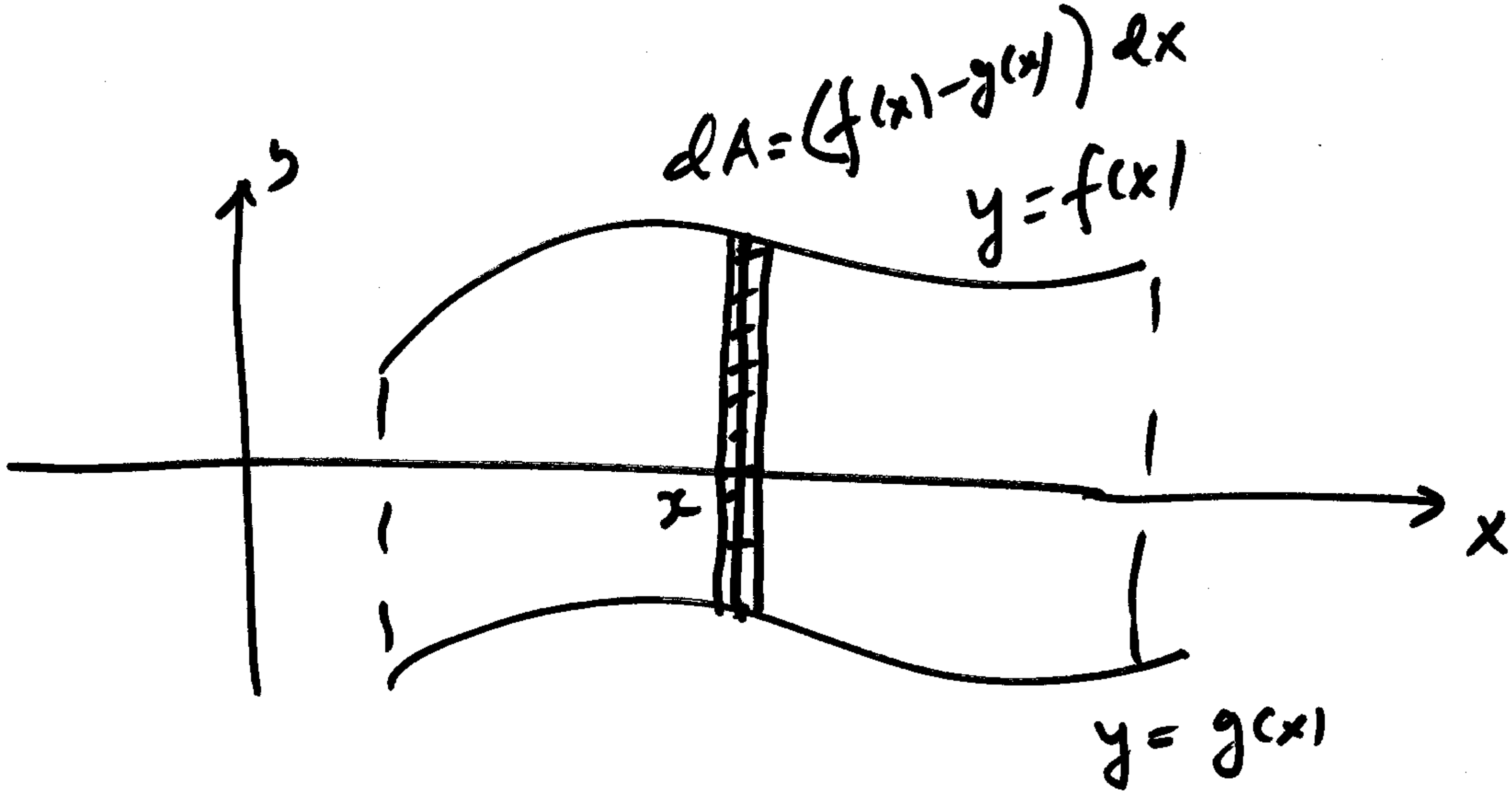


$A = \int_a^b f(x)\,dx$ = area under the graph of $f(x)$ above the x-axis for $a \leq x \leq b$.



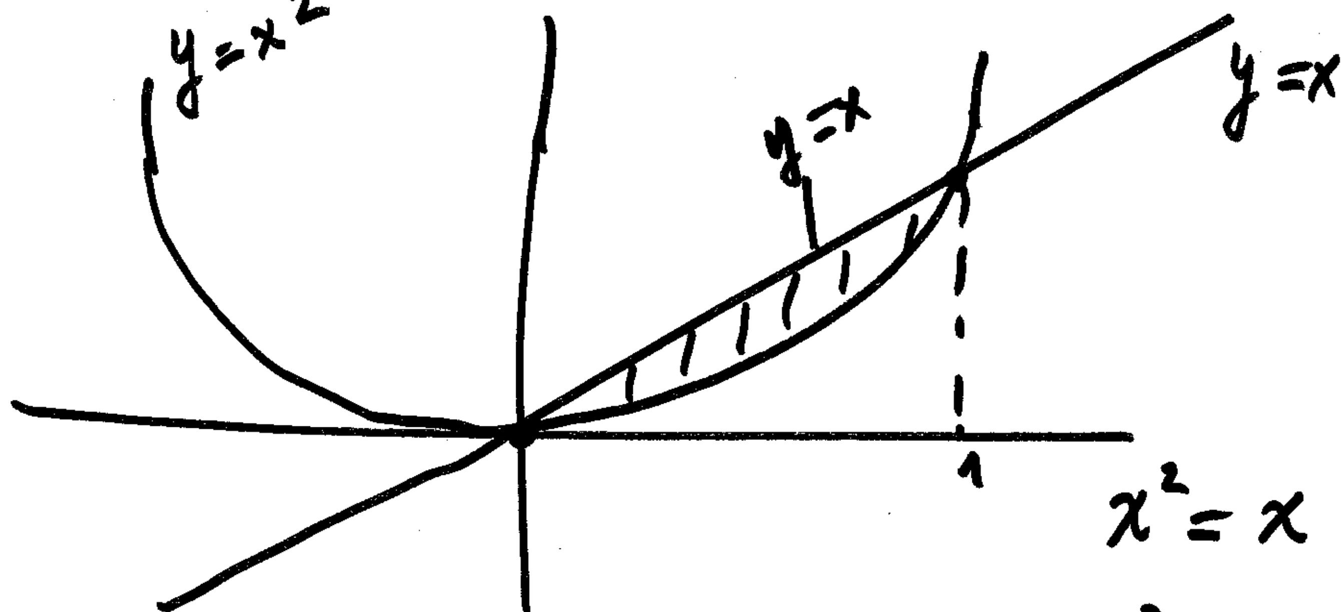
$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \left(f(x) - g(x)\right)dx$$

Area of the region bounded by the two curves.



$$A = \int_a^b \left(f(x) - g(x) \right) dx$$

Examples: Find the area of the region bounded by $y = x$ and $y = x^2$



$x^2 = x$

$x^2 - x = 0$

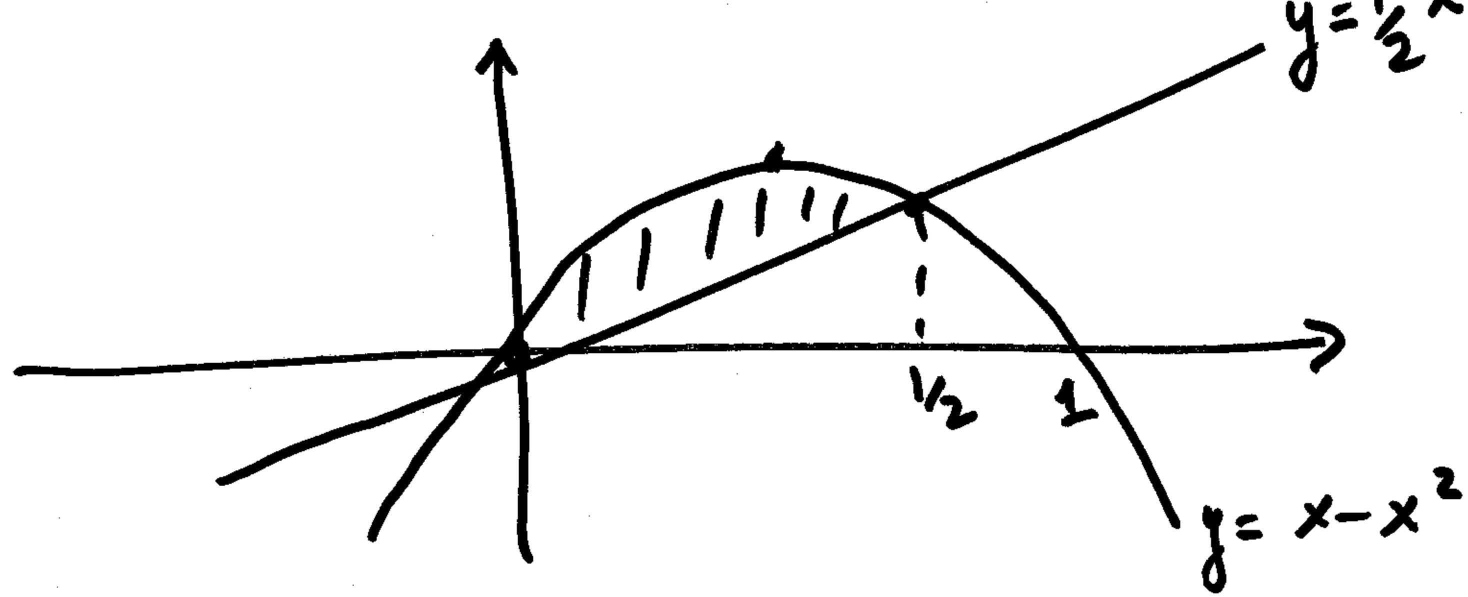
$x(1-x) = 0$

$x = 0, \quad x = 1$

$$A = \int_0^1 (x - x^2)\,dx$$

$$\int_0^1 (x - x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

2) $y = x - x^2, \quad y = x/2$



$x - x^2 = \frac{1}{2}x; \quad x/2 - x^2 = 0$

$x(1/2 - x) = 0$

$$A = \int_0^{1/2} \left(x - x^2 - \tfrac{1}{2}x\right) dx = \int_0^{1/2} \left(\tfrac{1}{2}x - x^2\right) dx$$

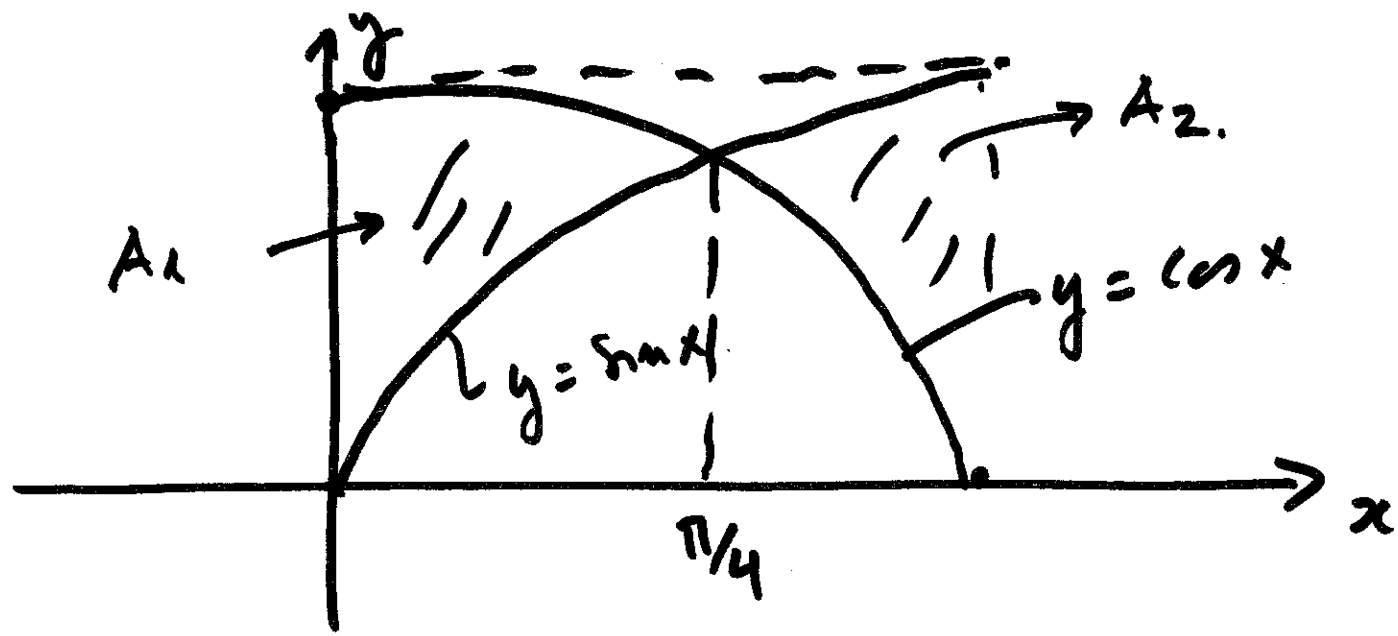
$$= \frac{x^2}{4} - \frac{x^3}{3} \Big|_0^{1/2} = \frac{1}{4}\cdot\frac{1}{4} - \frac{1}{3}\cdot\left(\frac{1}{2}\right)^3$$

$$= \frac{1}{16} - \frac{1}{24} = \frac{3-2}{48} = \frac{1}{48}.$$

3) $y = \cos x,\ y = \sin x$ $\quad x = 0$

and

$y = \cos x,\ y = \sin x,\quad x = \pi/2$.



$$A_1 = \int_0^{\pi/4} (\cos x - \sin x)\, dx$$

$$= (\sin x + \cos x)\Big|_0^{\pi/4} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)$$

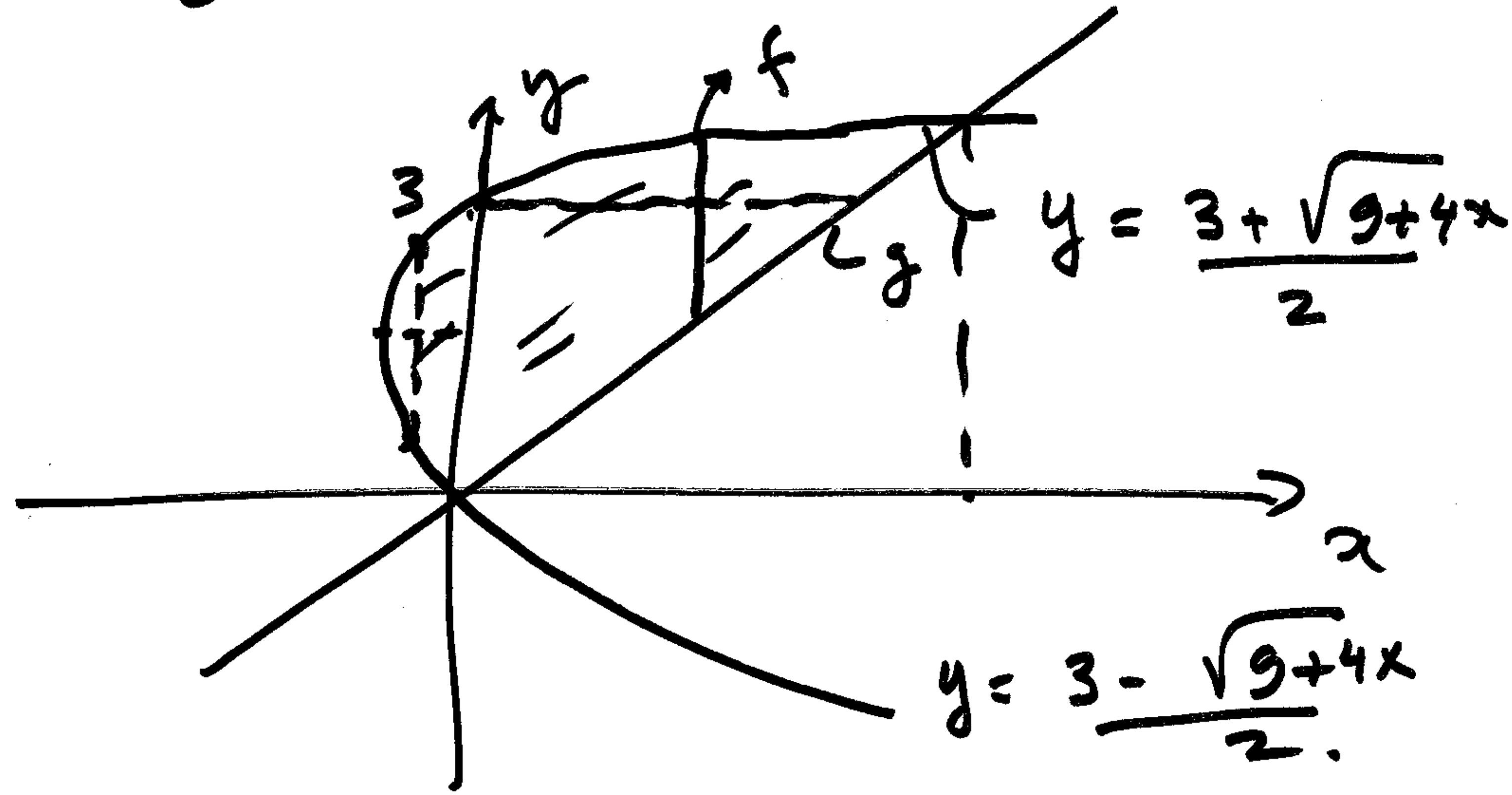
$$= \sqrt{2} - 1.$$

$$A_2 = \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx = -\cos x - \sin x \Big|_{\pi/4}^{\pi/2}$$

$$= -(\cos x + \sin x)\Big|_{\pi/4}^{\pi/2} = -\left(1 - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\right)$$

$$= -(1 - \sqrt{2}) = \sqrt{2} - 1.$$

4) $x = y^2 - 3y$; $x = y$.



<u>Solution 1</u>: (Bad idea, don't do it)

$$y^2 - 3y - x = 0$$

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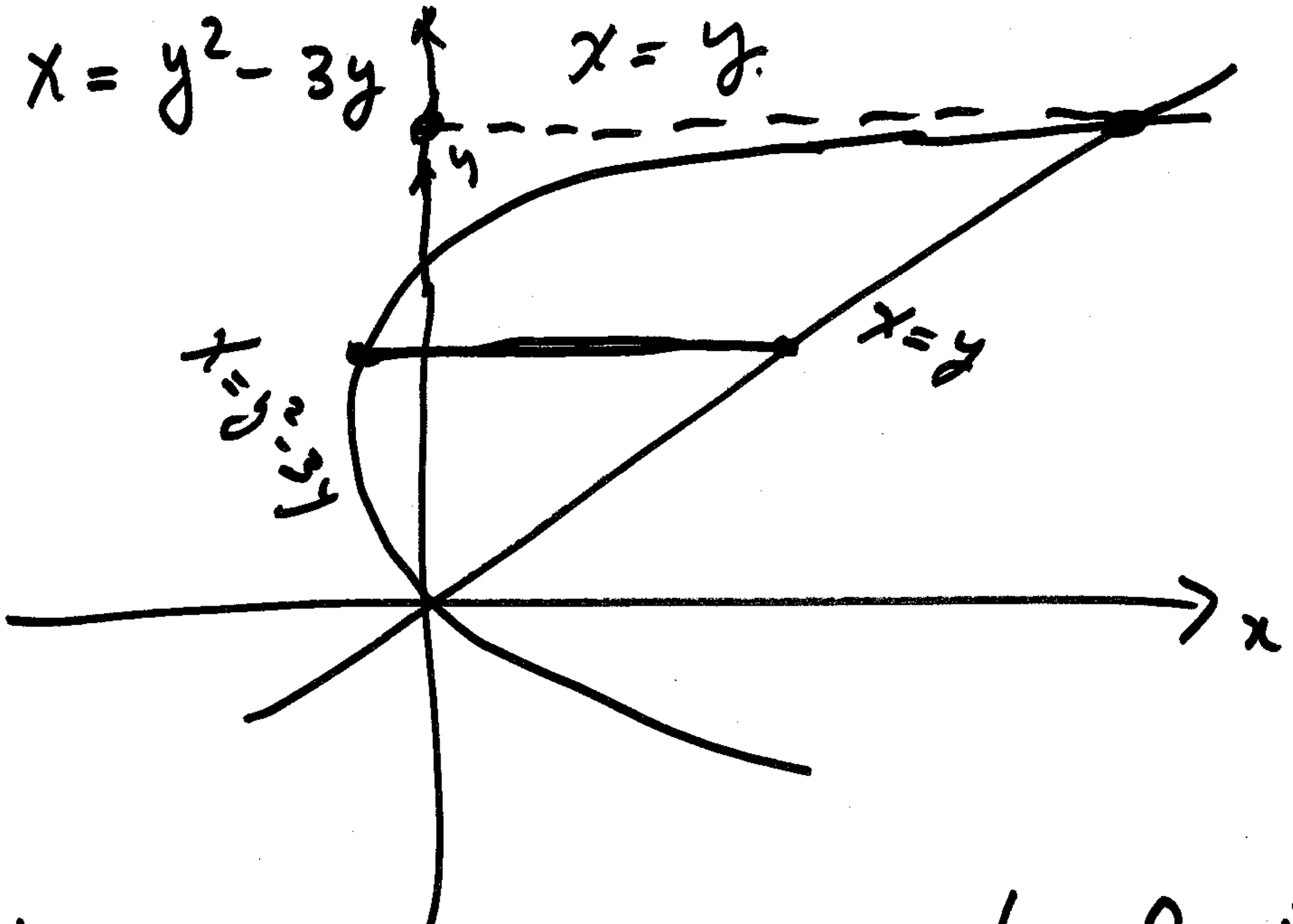
$$y = \frac{3 \pm \sqrt{9 + 4x}}{2}$$

$$\boxed{ay^2 + by + c = 0 \quad y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$A_2 = \int_0^4 \left(\frac{3 + \sqrt{9 + 4x}}{2} - x \right) dx$$

Solution 2 (the good one)



Write the area as an integral in the y variable.

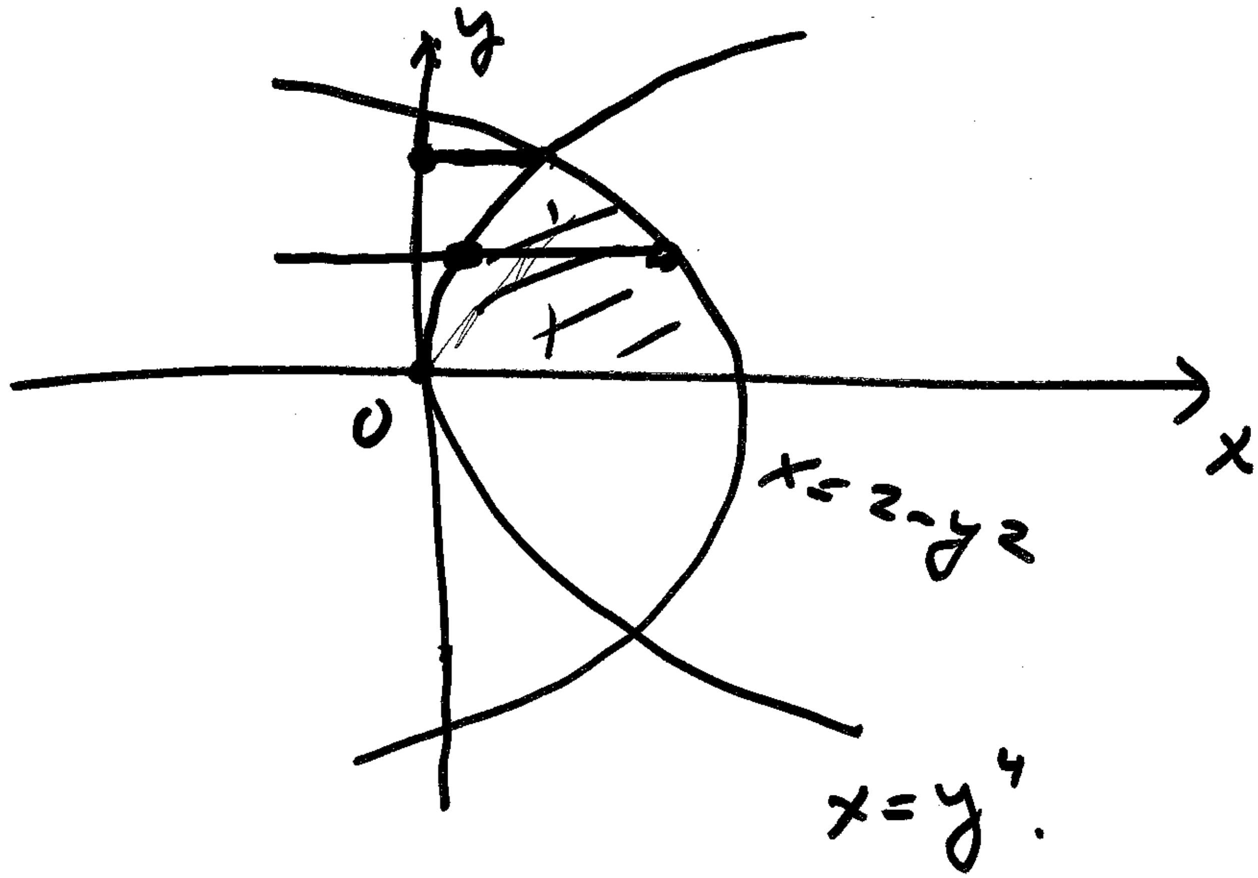
$$A = \int_0^4 \left(y - (y^2 - 3y)\right) dy$$

$$y^2 - 3y = y \quad ; \quad y^2 = 4y \qquad \begin{array}{l} y = 0 \\ y = 4 \end{array}$$

5) $x = y^4$, $y = \sqrt{2-x}$. and $y = 0$

$2 - y^2 = y^4$

$y = 1$



$y = \sqrt{2-x}$; $y^2 = 2 - x$; $x = 2 - y^2$.

$$A = \int_0^1 (2 - y^2 - y^4)\, dy$$

$$= 2 - \frac{1}{3} - \frac{1}{5} = \cdots$$